CAPITAL UNIVERSITY OF SCIENCE AND TECHNOLOGY, ISLAMABAD

# Fixed Point Theorems in Distance Spaces

by

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*Dedicated*

*to my beloved and dearest*

*parents and teachers*

*&*

*my family.*

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This is to certify that the research work presented in the thesis, entitled "**Fixed Point Theorems in Distance Spaces**" was conducted under the supervision of **Dr. Dur-E-Shehwar Sagheer**. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the **Department of Mathematics, Capital University of Science and Technology** in partial fulfillment of the requirements for the degree of Doctor in Philosophy in the field of **Mathematics**. The open defence of the thesis was conducted on **August 05, 2022**.

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# *List of Publications*

It is certified that following publication(s) have been made out of the research work that has been carried out for this thesis:-

1. **M. Anwar**, D. Shehwar, R. Ali, and N. Hussain, "Wardowski Type ($\alpha$,$F$)-Contractive Approach for Nonself Multivalued Mappings," University Politehnica of Bucharest Scientific Bulletin, A Series: Applied Mathematics and Physics, vol. 82, Issu, 1(2020), pp. 69-78, 2020.

2. M. U. Ali, T. Kamran, F. Din, **M. Anwar**, "Fixed and Common Fixed Point Theorems for Wardowski Type Mappings in Uniform Spaces," University Politehnica of Bucharest Scientific Bulletin, A Series: Applied Mathematics and Physics, vol. 80, Issu. 1(2018), pp. 3-12, 2018.

3. **M. Anwar**, D. Shehwar, R. Ali, "Fixed Point Theorems on ($\alpha$,$F$)-Contractive Mapping in Extended $b$-Metric Spaces," Journal of Mathematical Analysis, vol. 11, Issu. 2(2020), pp. 43-51, 2020.

4. **M. Anwar**, D. Shehwar, R. Ali, S. Batul, "Fixed Point Theorems of Wardowski Type Mappings in $S_b$-Metric Spaces," Thai Journal of Mathematics, vol. 20, Issu. 2(2022), pp. 945-956, 2022.

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# *Abstract*

A new notion of $(\alpha,F)$-contraction for multivalued mappings is introduced in this dissertation. Some fixed point results are established in this setting on the platform of metric spaces. These results are validated by providing suitable examples. An application is also provided to enhance the usefulness of the result. Similarly certain fixed point results are proved by introducing $(\alpha, F)$-contractive single valued mappings in the structure of uniform spaces. The structure of extended $b$-metric spaces is used to prove some fixed point results by introducing $(\alpha, F)$-contractive single valued mappings. This notion of $(\alpha, F)$-contractive single valued mapping is further generalized in the structure of $S_b$-metric spaces to provide some fixed point theorems. Certain suitable examples are given to validate these results. These results are generalizing many existing fixed point theorems and can be very helpful in computing the solutions of various problems related to physics and engineering, occurring in interdisciplinary research.

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# Abbreviations

| | |
|---|---|
| **BC** | Banach Contraction |
| **BCP** | Banach Contraction Principle |
| **CFP** | Common Fixed Point |
| **Common F P** | The set of all common fixed points for the pair of self mappings $T$ and $S$. |
| **Common F P Theorem** | Common fixed point Theorem for the pair of self mappings $T$ and $S$. |
| **EbMS** | Extended $b$-Metric Spaces |
| **F P** | Fixed Point |
| **F P Theorem** | Fixed Point theorem |
| **L.S.C** | Lower Semi-continuous |
| **MS** | Metric Spaces |
| **MVM** | Multivalued Mappings |
| **NS** | Nonself |
| **PIT** | Picard Iterative Theorem |
| **PO** | Picard Operator |
| **SVM** | Single Valued Mappings |
| $S_b$**MS** | $S_b$-Metric Spaces |
| **U.S.C** | Upper Semi-continuous |
| **US** | Uniform Spaces |

# Symbols

| | |
|---|---|
| $\mathbb{C}$ | Set of Complex Numbers |
| $CB(M)$ | Closed and Bounded Subsets of Set $M$ |
| $CL(M)$ | Closed Subsets of Set $M$ |
| $d$ | Distance between two points |
| $\delta$ | Distance between a point and a set |
| $D$ | Distance between two Sets |
| $H$ | Hausdorff Metric |
| inf | Greatest lower bound |
| sup | Least upper bound |
| max | Maximum |
| min | Minimum |
| $M_x$ | Maximum Distance |
| $\mathbb{N}$ | Set of Natural Numbers |
| $N(M)$ | Non-empty Subsets of Set $M$ |
| $P(M)$ | Power set of Set $M$ |
| $\mathbb{R}$ | Set of Real Numbers |
| $\mathbb{R}^+$ | Set of Positive Real Numbers |
| $\mathbb{R}_+ = [0, \infty)$ | Set of Non-Negative Real Numbers |

# Chapter 1

# Introduction

Functional analysis is a branch of classical mathematical analysis started in later part of 19th century and was ultimately accepted in the 1920's and 1930's. Functional analysis is one of the most interesting branch of mathematics that plays an essential role in many areas of applied sciences as well as in mathematics. This field of mathematical analysis has flourished very rapidly in many branches of mathematics. Functional analysis deals with functionals or functions of functions over infinite dimensional spaces. The evolution process of functional analysis methods were started almost a century ago.

The beautiful outcome of this development is appeared as fixed point theory. For many decades fixed point theory is considered a very productive and progressive outcome of nonlinear functional analysis. This theory is a striving and an emerging area for research and development and it is, in fact, a smart combination of different disciplines of knowledge like Geometry, Topology and Analysis.

In various disciplines of mathematics and applied sciences, the issue of existence of a solution in nonlinear problems has remained to be an important topic. Fixed point theory assures the existence of a solution of nonlinear problems, by proving the existence of fixed point. Some initial level work on fixed point theory was initiated in 1866, by Poincare [1] and he may rightly be considered as a pioneer as he gave his first fixed point theorem without its proof in 1883-1884. Brouwer [2] was the first man who proved a fixed point theorem on unit sphere in 1912 and it is stated as one of the early approaches that was further explored by Kakutani [3]. It is worth to be mentioned here that the fixed point theory is used as a technique of successive approximation which helps us in three ways:

1. to establish the fact of existence of solution for the nonlinear problems;
2. to establish the fact that nonlinear problems have a unique solution;
3. to establish an iterative scheme and conclude that the fixed point is exactly the limit of the iterative sequence.

The appearance of the fixed point theory that started in the later part of the nineteenth century, was used for successive approximations to search out the existence and detecting a unique solution of differential equations. This approach and methodology is linked with the names of eminent mathematicians such as Liouville, Lipschitz, Cauchy, Peano, Fredholm and particularly, Picard. Actually the primary concepts related to the fixed point theoretic and systematic methodological approaches are clearly visible in the prominent works of Picard. However, Banach [4], a prominent mathematician, did creditable work by placing the underlying concepts into an abstract structure. This work is more adjustable for broad based utilization and better than the ones which are applicable only for elementary differential and integral equations. Fixed point theory has a long historical background of wide range applications in many different disciplines of pure as well as applied mathematics [5–34]. Although it is a well saturated field but a process of research continuity is showing that it is still an active and open area.
The most vital and significant role of this theory is related to the existence of the solution of different operator equations which have certain attributes, for instance the solution of linear integral equations, such as, Fredholm integral equations and Volterra integral equations that has the useful impact for the promotion and the development of the modern and versatile ideas. The fixed point theory is also applied in many areas like computer sciences, medical diagnosis, neural network, artificial intelligence and in many other relevant fields as well. There are important areas where the applications of fixed point theory are very effective such as

in boundary values and eigenvalues problems. Fixed point theory is, in fact, a classical approach in the finding of approximate solution and to verify the system stability.

In such an abstract approach, in general, one usually begins its work with a set of elements that are satisfying certain axioms. In this technique the nature of the elements is generally kept unspecified and it can be done on and when required. The theory then consists of logical consequences that conclude from the axioms, which are once and for all derived as theorems. It implies that in this axiomatic manner one frames a mathematical model whose theory is evolved in a theoretic approach. Later on, these general theorems can widely be utilized in various particular sets that satisfy such axioms. For instance, in algebra this approach is used in connection with fields, rings and groups.

The major development in fixed point theory is in two main directions. One direction of generalizations of fixed point results is, in the context of distinct spaces, for examples, metric spaces, Banach spaces, Hilbert spaces, topological spaces and even by changing in the structures of spaces [11, 13–23] where as the second direction is the conditions of contractions [24–37].

The idea of metric space was presented by Frechet [38] in 1906 who may rightly be called the founder of metric space and he indeed presented this notion of metric in an axiomatic manners as a generalized formulation of the Euclid distance. On the other hand, the concept of Hausdorff distance is due to Hausdorff [39]. It is noticed that the notion of the metric is so vital as it plays a central role in the fields of real analysis, complex analysis and functional analysis. Taking into account the central role of this notion in mathematics and fundamental sciences, it is extended and generalized in many distinct directions.

In this chapter, only a short list is given for the interest of readers as it is not possible to discuss here all these notions which are available in the literature. After this development, researchers have setup many new ways and means for generalizing this space as several versions, adaptations, generalizations, and extensions of metric in the forms of 2-metric, cone metric, $D$-metric, $G$-metric, modular metric, quasi metric, multiplicative metric, ultra metric, $b$-metric, dislocated metric, symmetric metric, Hausdorff metric, $S$-metric, partial metric, setvalued metric, fuzzy metric, and many more metrices or the metric spaces, for instance, see these references [12, 31–34, 37, 40–73].

These generalizations are established by changing, modifying, adding and subtracting properties and conditions of metric spaces. One of the emerging and most interesting generalization of a notion of metric, namely $b$-metric. We can see that in last few decades many new structures are designed by mathematicians, for examples, $b$-metric spaces [12, 33, 34, 53, 54, 56–61, 66, 74], rectangular metric spaces [62, 63], rectangular $b$-metric spaces [71] and extended $b$-metric spaces [64–70].

In 1989, Bakhtin [53] has given a new concept of $b$-metric spaces after many decades. It may rightly be called a first generalization of metric space. He achieved this target by changing the triangular inequality of metric space. This notion of $b$-metric space was further extended by Czerwik [33, 37] as a generalization of the metric space by using the weaker triangular condition. Later on, Czerwik [31, 34] used these concepts of $b$-metric spaces and presented some results related to fixed points. Some preliminary work on this notion was initiated by Bourbaki [23] and Bakhtin, which are based on Banach contraction. Some further work was also extended in this regard by Dikranjan [11] and Heinonen [21]. In this setting some new results are proved using complete $b$-metric space structure for single valued mappings [12, 58, 59, 67–69] and later on, to more general mappings, for examples, multivalued mappings or set valued mappings [60, 61, 75]. The notion of $b$-metric space was further generalized in 2000, by Branciari [76] which is named as rectangular metric space.

In 2015, George *et al.* [71] introduced rectangular $b$-metric space and they proved some fixed point results. Another important generalization of $b$-metric space known as EbMS is given by Kamran *et al.* [70] in 2017. Authors proved some fixed point results endowed with EbMS [64–70].

Weil, in 1937, presented the concept of uniform spaces in its explicit form for the first time but before this some formal concepts related to uniform spaces were prevailing. Dikranjan [11] presented the concept of uniform space by defining entourages in Topologie Generale and Tukey [77], in 1940, provided the concept of uniform cover. Weil [78] also presented this notion as a family of pseudo-metrics. An interesting concept of cone metric space was given by Huang *et al.* [72], in 2007. This is an appealing and wide-range expanding role of metric space in which metric is repealed by a function with images are taken in the structure of an ordered Banach space. Huang *et al.* contributed in the idea of definition to prove the properties of sequence in cone metric space. Later on, certain fixed point results for contractive single valued and multivalued mappings are obtained by using this new approach. Ma *et al.* [74] introduced a noticeable generalization of cone metric spaces known as $C^*$-algebra valued metric spaces and provided generalized Banach contraction theorems by using this approach. $C^*$-algebra valued metric spaces have become an exciting topics for researchers now a days (see [73], [79], [80]).

In the second direction, the researchers adopted many other ways of generalizations by changing the incredible Banach contraction into different structures. Several attempts have been made in this direction and many interesting conditions on the mapping are imposed to find the existence of fixed point. Banach [4] used this contraction in an outstanding way to prove the existence of fixed point in his eminent Banach contraction principle (BCP). This principle [4] states that every self-mapping $T$ on a complete metric space $(M, d)$, satisfying:

$$d(Tm_1, Tm_2) \leq \lambda d(m_1, m_2), \ \ \forall \ \ m_1, m_2 \in M \ \text{ and } \ \lambda \in [0, 1),$$

has a unique fixed point.

Banach contraction principle appeared for the first time as in its explicit form, in the Banach's PhD thesis work [4]. It is rightly stated that it was a milestone towards the solution of nonlinear problems in different disciplines of science and technology, as it provides a certain and a reliable process regarding the existence of solution and a strong technique of convergence of different iterative schemes in numerical methods. The fact of significance of Banach contraction principle, in general, is lying in the reality that the under consideration space is complete in nature and it includes the error estimates. Edelstein [81] was the person who introduced an important generalization of the Banach contraction condition by taking constant $\lambda = 1$ and using distinct points from the space $M$, in 1962. In the same year Rakotch [82] introduced another contractive condition, replacing the constant $\lambda$ by a monotonic decreasing function $\lambda : [0, \infty) \to [0, 1]$. This contraction is as follows:

$$d(Tm_1, Tm_2) \leq \lambda(t) d(m_1, m_2),$$

for all $m_1, m_2 \in M$.

In 1968, Kannan [36] presented a contraction condition that does not imply the continuity like Banach contraction. This contraction condition is:

$$d(Tm_1, Tm_2) \leq \lambda\{d(m_1, Tm_1) + d(m_2, Tm_2)\},$$

for all $m_1, m_2 \in M$ and $0 < \lambda < \frac{1}{2}$.

In 1969, Kannan [30] proved another extension of Banach contraction principle. Following the work of Kannan [30, 36], Chatterjea [29] introduced another type of contraction for the existence of fixed point in 1972, which is as follows:

$$d(Tn_1, Tn_2) \leq \lambda\{d(n_1, Tn_2) + d(n_2, Tn_1)\},$$

for all $n_1, n_2 \in M$ and $0 < \lambda < 1$.

A massive research has been done in this direction by many authors, for examples, Bianchini [32], Hardy [24], Reich [27, 35], Ciric [25] and Caristi [26] by using different conditions on the mappings under consideration. A comprehensive comparison of different contractions is presented by Rhoades [28], in 1977.

Nadler [83] is the founder of set valued contraction and he laid the foundation of fixed point results in the setting of multivalued mappings. The occurrence of multivalued mappings can be observed in the first quarter of 20th century. Nadler [83] introduced the multivalued contractive mappings. He proved two fixed point theorems for another notion of multivalued contractive mappings. The first theorem is the generalization of Banach contraction principle in which it was proved that a multivalued contraction mapping has a fixed point that is defined on complete metric space into a nonempty closed and bounded subset of metric space. The second theorem is the generalization of Edelstein result for compact set-valued local contractions.

A new concept related to cyclic contraction notion was presented by Kirk *et al.* [84]. It is worth mentioning here that the cyclic contraction are not to continuous and this fact was used as an advantage in proving many results [10, 75, 85–93]. Some other results are produced by taking the interesting new notions of $(\alpha,\psi)$-admissible mappings [10, 85, 89, 90, 93, 94] and this notion was introduced by Samet *et al.* [90], in 2012.

In 2012, Wardowski [95] presented another well known contraction, $F$-contraction. Sagroi *et al.* [5] proved fixed point results on $F$-contraction in 2013 with some applications on integral equations. A lot of work is presented in this direction, see for examples, [8, 9, 12, 52, 96–99]. $F$-contraction was further generalized in many ways.

An interesting generalization of $F$-contraction is $(\alpha,F)$-contractive mapping. $(\alpha,F)$-contractive mapping was firstly introduced by Kamran *et al.* [12] in 2016 in the structure of $b$-metric space on single valued mappings. In 2017, this contraction was further extended to multivalued by Hussain *et al.* [100].

This dissertation includes the research work that is obtained by establishing $(\alpha,F)$-contractive mappings and opting the idea of Ali *et al.* [75] in which they proved some results on nonself mapping in 2014, Kamran *et al.* [70] in which they presented their fixed point results by introducing a new space, named as, an extended $b$-metric space in 2017 and Souayah *et al.* [101] in which they produced some new interesting fixed point results on the new platform of $S_b$-metric spaces in 2016. Our work on $(\alpha,F)$-contractive mapping is [7, 48, 102–104].

The organization of thesis is given below:

Chapter 2 is focused on some fundamental and basic concepts which are used in subsequent chapters to present main contributions beautifully. The major aim of this chapter is to focus on relevant literature and to review and refurnish these concepts. Different types of mappings are elaborated with some examples. Some important generalizations of BCP are provided without including their proofs. Proofs can be found in the given references.

This chapter also includes different types of contractions.

In the chapter 3, a contraction using the idea of $F$-function and $\alpha$-mapping is established. This is named as $(\alpha,F)$-contraction on multivalued mappings. Some fixed point results are proved by using $(\alpha,F)$-contraction. Examples along with an application are also provided.
This work is published as:

**M. Anwar**, D. Shehwar, R. Ali and N. Hussain, "Wardowski type $(\alpha,F)$-contractive approach for nonself multivalued mappings," University Politehnica of Bucharest Scientific Bulletin, A Series: Applied Mathematics and Physics, vol. 82, issu. 1(2020), pp. 69-78, 2020.

Chapter 4 addresses a new Wardowski type contraction by combining $\alpha$-mappings and $F$-function, namely, $(\alpha,F)$-contraction on self mapping in uniform spaces. Certain fixed point and common fixed point theorems are established in this setting. A few examples are also presented for the validation of these results.
This work is also published as:

M. U. Ali, T. Kamran, F. Din and **M. Anwar**, "Fixed and Common Fixed Point Theorems for Wardowski Type Mappings in Uniform Spaces," University Politehnica of Bucharest Scientific Bulletin, A Series: Applied Mathematics and Physics, vol. 80, Issu. 1(2018), pp. 3-12, 2018.

Chapter 5 contains some fixed point results proved by using the platform of extended $b$-metric spaces accompanied with $(\alpha,F)$-contraction.
This research appeared in literature as:

**M. Anwar**, D. Shehwar and R. Ali, "Fixed Point Theorems on $(\alpha,F)$-contractive Mapping in Extended $b$-Metric Spaces," Journal of Mathematical Analysis, vol. 11, Issu. 2(2020), pp. 43-51, 2020.

In the chapter 6, certain significant results related to fixed point and common fixed point are established in $S_b$-metric spaces by using $(\alpha,F)$-contraction.
This research work published in literature as:

**M. Anwar**, D. Shehwar, R. Ali and S. Batul, "Fixed Point Theorems of Wardowski Type Mappings in $S_b$-Metric Spaces," Thai Journal of Mathematics, vol. 20, Issu. 2(2022), pp. 945-956, 2022.

In chapter 7, we have concluded our research study. We have discussed the targets we achieved and planned for future.

# Chapter 2

# Preliminaries

This chapter is focused on some primary concepts which are necessary for the purpose to lay down a strong base for this dissertation. These fundamental themes are very vital for the better presentation and understanding of this research work. The major aim of this chapter is to focus on relevant literature without including the formal proofs of the theorems.

## 2.1 Some Tools from Analysis

In this section some basic and important tools are selected from real analysis. It also includes some important types of mappings for the better understanding of the subject.

**Definition 2.1.1.**

"Let $M$ be a given nonempty set. The relation $\preceq$ is said to be a partial order relation on the given set $M$ if the following statements hold for each element $m, n, t \in M$:

1. $m \preceq m$; (Reflexive)
2. $m \preceq n$ and $n \preceq m \Leftrightarrow m = n$; (Anti-symmetric)

3. $m \preceq n$ and $n \preceq t \Rightarrow m \preceq t$. (Transitive)

The set $M$, whose elements satisfy above properties is called a partially order set. That is, $M$ is said to be partially ordered if each pair of elements of $M$ is not related by a certain order.
If all the elements of a set $M$ are comparable under an order $\preceq$, then the set $M$ is called a totally ordered set with respect to the order $\preceq$" [105].

The following examples will elaborate the above idea transparently:

**Example 2.1.1.**

1. Consider the set of real numbers $\mathbb{R}$. Set of reals is totally ordered set for the usual ordering $\leq$ of the real numbers.

2. Consider that $P(M)$ is a power set of a given nonempty set $M$ and a relation $\preceq$ is given by the inclusion relation.
   We say $C \preceq D$ if $C \subset D$, where $C, D \in P(M)$. One can easily check that $\preceq$ is a partial order and $P(M)$ is partially ordered set.

3. Consider
   $$M = \mathbb{R} \times \mathbb{R} = \{(m_1, m_2) : m_1, m_2 \in \mathbb{R}\}.$$
   Define an order $\preceq$ on the set $M$ in the following way:
   $$(m_1, m_2) \preceq (n_1, n_2) \Leftrightarrow m_1 \leq n_1, m_2 \leq n_2.$$
   Here $\leq$ is the usual order on the elements of $\mathbb{R}$. Then it can be seen easily that $\preceq$ is a partial order on the given set $M$ or $M$ is a partially ordered set.

**Definition 2.1.2.** Let $M$ be a nonempty subset of $\mathbb{R}$ and $T : M \to \mathbb{R}$ be a real valued function. Then the limit supremum of mapping $T$ for $\epsilon > 0$, is defined in the following way:

$$\limsup_{n \to m} T(n) = \begin{cases} \sup\{T(n) : |m-n| < \epsilon\}; & \text{if the supremum exists,} \\ \infty; & \text{otherwise,} \end{cases}$$

Let $M$ be a nonempty subset of $\mathbb{R}$ and $T : M \to \mathbb{R}$ be a real valued function. Then the limit infimum of mapping $T$ for $\epsilon > 0$, is defined in the following way:

$$\lim_{n \to m} \inf T(n) = \begin{cases} \inf\{T(n) : |m - n| < \epsilon\}; & \text{if the infimum exists,} \\ -\infty; & \text{otherwise.} \end{cases}$$

The concept of infimum and supremum can be elaborated in simple words as follows:

Consider a nonempty set of real numbers $M$ which is bounded from below. A number $m_0 \in M$ is called the infimum of set $M$ if $m_0$ is a greatest lower bound of $M$. It can be expressed as:

$$m_0 = \inf M.$$

In similar way:

Consider a nonempty set of real numbers $M$ which is bounded from above. A number $m_0 \in M$ is called the supremum of set $M$ if $m_0$ is a least upper bound of $M$. It can be expressed as:

$$m_0 = \sup M.$$

Some useful observations about the supremum and infimum are given below:

1. If a subset $M$ of real numbers $\mathbb{R}$ is bounded below then this subset has an infimum.
2. If a subset $M$ of real numbers $\mathbb{R}$ is bounded above then this subset has a supremum.

**Example 2.1.2.**
Let $m_t$ be the sequence $\left\{\frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, ....\right\}$.

Now we consider that the $\{a_t\}$ and $\{b_t\}$ are the sequences of the infimum and supremum of the sub-sequences:

$$\{a_t\} = \left\{-\frac{1}{2t+1}\right\}, \quad \text{and} \quad \{b_t\} = \left\{\frac{1}{2t}\right\}, \qquad \text{where,} \qquad t \in \mathbb{N}.$$

The infimum of $m_t$ is the greatest lower bound of all the sub-sequences of $\{a_t\}$ and the supremum of $m_t$ is the least upper bound of all the sub-sequences of $\{b_t\}$ respectively. Therefore,

$$\limsup_{t\to\infty} m_t = \lim_{t\to\infty}\left\{\frac{1}{2t}\right\} = 0, \quad \text{and} \quad \liminf_{t\to\infty} m_t = \lim_{t\to\infty}\left\{-\frac{1}{2t+1}\right\} = 0.$$

Thus, the limit of both the sequences i. e. $\{a_t\}$ and $\{b_t\}$ is 0 as $t \to \infty$, which is also the limit of sequence $m_t$ i. e. $\lim_{t\to\infty} m_t = 0$.

Before introducing the formal concepts of left and right continuity, we recall the concept of metric space.

The idea of using abstract space in a systematic manner is first given in 1906 by Frechet [38] and it is justified by its usefulness in different fields of mathematics. Metric space is a fundamental and basic concept in functional analysis and this space behaves same as the line $\mathbb{R}$ in calculus. In fact metric space generalize the idea of distance between points of $\mathbb{R}$.

To achieve the certain goals of the research in this dissertation, the concept of metric space is given below.

**Definition 2.1.3.**

“Let $M$ be a nonempty set and $d\colon M \times M \to [0,\infty)$ be a function which satisfies the following properties for all $m_1, m_2, m_3 \in M$:

(M1) $d$ is real-valued, finite and non-negative, **(Non-negativeness)**

(M2) $d(m_1, m_2) = 0$ if and only if $m_1 = m_2$, **(Identification)**

(M3) $d(m_1, m_2) = d(m_2, m_1)$, **(Symmetry)**

(M4) $d(m_1, m_2) \leq d(m_1, m_3) + d(m_3, m_2)$. **(Triangle inequality)**

Then $d$ is called metric on $M$ and the pair $(M, d)$ is called a metric space” [18].

The property (M2) is so important as it provides a guarantee for the uniqueness of limit of a sequence along with the property (M4). If condition (M4) is omitted then such space is called semi-metric space. In the semi-metric space limit of a sequence is not unique and convergent sequence need not to be Cauchy. The property (M4) is also known as sub-additive property and it is taken from elementary geometry which states that in a triangle the sum of the any two lengths of a triangle is always greater than the third side.

**Example 2.1.3.**

Consider a set $M = C[a,b]$, the set of all bounded and continuous functions. Let the metric $d\colon M \times M \to [0,\infty)$ is defined as given below:

$$d(m,n) = \int_a^b \Big|m(t) - n(t)\Big| dt; \quad \forall\, m(t), n(t) \in M.$$

The first three properties are easy to prove. To prove the triangular inequality, we proceed as follows:

$$\begin{aligned} d(m,n) &= \int_a^b \Big|m(t) - n(t)\Big| dt = \int_a^b \Big|m(t) - y(t) + y(t) - n(t)\Big| dt \\ &\leq \int_a^b \Big|m(t) - y(t)\Big| dt + \int_a^b \Big|y(t) - n(t)\Big| dt = d(m,y) + d(y,n). \end{aligned}$$

Hence $d$ is metric on $M$ as all the conditions of metric space are satisfied and the pair $(M, d)$ is a metric space.

**Example 2.1.4.**

Consider that $\ell^\infty$ be the set of all bounded real or complex sequences. Define a metric function $d\colon \ell^\infty \times \ell^\infty \to [0,\infty)$ as given by:

$$d(m,n) = \max_{t\in\mathbb{N}}\{|m_t - n_t|\}\ ; \quad \forall\ m,n \in \ell^\infty \quad \text{where}\ ; \qquad m = \{m_t\} \quad \text{and} \quad n = \{n_t\}.$$

The first three properties are easy to prove. To see the triangular inequality, we proceed as follows:

$$\begin{aligned} d(m,n) &= \max\{|m_t - n_t|\} \\ &= \max\{|m_t - x_t + x_t - n_t|\} \\ &\leq \max\{|m_t - x_t|\} + \max\{|x_t - n_t|\} \\ &\leq d(m,x) + d(x,n). \end{aligned}$$

Thus $(\ell^\infty, d)$ is a metric space.

**Example 2.1.5.**

Let $M = \{1,2,3\}$. Define $d\colon M \times M \to [0,\infty)$ by:

$$d(m,n) = (m-n)^2; \quad \forall\, m,n \in M.$$

This function is not a metric.

$$d(3,1) = (3-1)^2 = 2^2 = 4, \text{ but } d(3,2)+d(2,1) = (3-2)^2+(2-1)^2 = 1^2+1^2 = 2.$$

Since in this set the triangular inequality is not satisfied.
So, $d$ is not a metric on $M$.

Completeness is another vital term which a metric space structure may or may not have. This property has some consequences which make it more important and prominent than the incomplete ones. This property does not follow from the properties of metric space.

**Definition 2.1.4.**

"A sequence $\{m_t\}$ in a metric space $M = (M,d)$ is said to converge or to be convergent if there is a point $m \in M$ such that $\lim\limits_{t\to\infty} d(m_t, m) = 0$, $m$ is called the limit of $\{m_t\}$ and we write $\lim\limits_{t\to\infty} m_t = m$ or, simply, we say that $\{m_t\}$ converges to $m$ or has the limit $m$. If $\{m_t\}$ is not convergent, it is said to be divergent" [18].

Let us recall from the fundamentals of real analysis that a sequence $\{m_t\}$ of real numbers is convergent in the real line $\mathbb{R}$ as well as a sequence $\{m_t\}$ of complex numbers is convergent in the complex plane $\mathbb{C}$ if and only if it satisfies the Cauchy criterion for convergence, that is, for every number $\epsilon > 0$, there is a number $N = N(\epsilon)$ such as $|m_t - m_s| < \epsilon;$ for; $t,s \geq N$. Here $|m_t - m_s|$ is the distance between two points $m_t$ and $m_s$ in the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$.
Generally, this is not true as there are Cauchy sequences which do not converge.
This discussion motivates to discuss the following concept regarding the completeness that was presented firstly by Frechet [38], in 1906.

**Definition 2.1.5.**

"A sequence $\{m_t\}$ in a metric space $M = (M, d)$ is said to be a Cauchy if for every $\epsilon > 0$ there exists a positive integer $t_0$ such that for all $t, s \geq t_0$, we have $d(m_t, m_s) < \epsilon$ or $d(m_t, m_s) \to 0;$ as $t, s \to \infty$" [18].

**Definition 2.1.6.**

"A metric space $(M, d)$ is said to be complete if every Cauchy sequence in $M$ converges to a point in $M$" [18].

The most well known examples of complete metric spaces are the set of real numbers and the set of complex numbers. Criteria of completeness for $M \subseteq \mathbb{R}$ is:

1. $M$ is complete $\Leftrightarrow$ $M$ is closed.
2. $M$ is compact $\Leftrightarrow$ $M$ is closed and bounded.

A set with discrete metric is a trivial example of a complete metric space. Here are some incomplete spaces.

1. $\mathbb{R} - \{0\}$; set of all real numbers except zero.
2. $\mathbb{Q}$; set of rational numbers.
3. $(a, b)$; open intervals.

## 2.2 Some Important Mappings

Fixed point theorems are predominately concerned with obtaining conditions on the structures and underlying spaces. It also deals with attributes of self mapping $T$ on $M$ for the purpose to get the extensions of fixed point results. Certain useful conditions on mappings are discussed in this section of the chapter.

(a) "Let $(M, d)$ be a metric space. A mapping $T\colon M \to M$ is said to be Lipschtizian if there is a constant $k \geq 0$ such that for all $m, n \in M$,

$d(Tm, Tn) \leq kd(m, n)$. The smallest number $k$ for which above inequality holds is called the Lipschtizian constant of $T$" [16]. Lipschtizian is a continuous mapping.

(b) "Let $(M, d)$ be a metric space. A mapping $T\colon M \to M$ is said to be contraction mapping if for every $m, n \in M$, $d(Tm, Tn) \leq kd(m, n)$, with $0 \leq k < 1$. This mapping is also known as Banach contraction" [16].

(c) "Let $(M, d)$ be a metric space. A mapping $T\colon M \to M$ is said to be contractive mapping if for every $m, n \in M$, $d(Tm, Tn) < d(m, n)$, with $m \neq n$" [16].

(d) "Let $(M, d)$ be a metric space. A mapping $T\colon M \to M$ is said to be non-expansive mapping if for every $m, n \in M$, $d(Tm, Tn) \leq d(m, n)$" [16].

One can easily conclude that:

1. Every contraction, contractive and non-expansive is a Lipschtizian mapping.
2. Every contraction and contractive is a non-expansive mapping.
3. Every contraction is a contractive mapping.

## 2.3 Hausdorff Metric Space

The distance between two closed sets, now a days, is used as a fundamental tool in mathematics, computer science and other interdisciplinary research. It was introduced more than one hundred year ago, in 1905, by Pompeiu [106] (1873-1954), and thereafter established in general setting of metric space and largely disseminated by Hausdorff since 1914. Pompeiu actually needed this distance in order to rigorously define the distance between two curves in the complex plane and also to introduce, by means of this distance, the concept of limit of sequence of sets.

Currently, in fixed point theory, this concept is adhered to see the existence of fixed point for multivalued mappings.

This section is dedicated to define Pompieu Hausdorff distance.

**Definition 2.3.1.**

"Let $(M, d)$ be a metric space where $M$ is a nonempty set. For any point $m_1 \in M$ and $B \subseteq M$, the distance between $m_1$ and the subset $B$ is defined as $\delta(m_1, B) = \inf\{d(m_1, m_2) : m_2 \in B\}$. We denote the class of all nonempty subsets of $M$ by $N(M)$, the class of all nonempty bounded closed subsets of $M$ by $CB(M)$ and $H$ is used as Hausdorff metric $H : CB(M) \times CB(M) \to [0, \infty)$ such that

$$H(A, B) = \max\left\{\sup_{m_1 \in A} \delta(m_1, B), \sup_{m_2 \in B} \delta(m_2, A)\right\}$$

for all nonempty subsets $A, B \in CB(M)$. Then $(CB(M), H)$ is called a Hausdorff metric space and $H$ is a Hausdorff metric" [96].

**Definition 2.3.2.**

"Let $(M, d)$ be a complete metric space for a nonempty set $M$ and a metric $d$. Consider a self mapping $T : M \to M$. For any point $m_1 \in M$ and a subset $B$ of $M$, the metric is defined as given $\delta(m_1, B) = \inf\{d(m_1, m_2) : m_2 \in B\}$. We denote the class of all nonempty subsets of $M$ by $N(M)$, the class of all nonempty closed subsets of $M$ by $CL(M)$ and $H$ is used as Hausdorff metric space $H : CL(M) \times CL(M) \to [0, \infty)$ such that

$$H(A, B) = \begin{cases} \max\left\{\sup\limits_{m \in A}\delta(m, B), \sup\limits_{n \in B}\delta(n, A)\right\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

for all nonempty subsets $A, B \in CL(M)$.

Then $(CL(M), H)$ is called a Generalized Hausdorff metric space" [96].

**Example 2.3.1.**

Consider the metric space $(\mathbb{R}, d)$, where $d$ is the usual metric on $\mathbb{R}$. Consider $A = [1, 23]$ and $B = [25, 40]$ are the subsets of $\mathbb{R}$. We find the Hausdorff distance between the set $A$ and the set $B$. Here the Hausdorff distance between the set $A$

and the set $B$ is defined by:

$$H(A,B) = \max\{\sup_{a\in A} \delta(a,B), \sup_{b\in B} \delta(A,b)\}.$$

Whereas the infimum distance for any point $a \in \mathbb{R}$ and a subset $B$ of $\mathbb{R}$ is given by $\delta(a,B) = \inf\{d(a,b) : b \in B\}$ and the supremum distance between two given subsets $A$ and $B$ of $M$ is given by $D(A,B) = \sup_{a\in A}\{\delta(a,B) : b \in B\}$.
Let $a \in A$ and $a = 13$, so

$$\delta(13,B) = \inf\{d(13,25) : 25 \in B\} = d(13,25) = |13-25| = 12.$$

Now

$$\begin{aligned} D(A,B) = D([1,23],[25,40]) &= \sup\{\delta(1,[25,40]), \delta(23,[25,40])\} \\ &= \sup\{d(1,25), d(23,25)\} = |1-25| = 24. \end{aligned}$$

Again

$$\begin{aligned} D(A,B) = D([1,23],[25,40]) &= \sup\{\delta([1,23],25), \delta([1,23],40)\} \\ &= \sup\{d(23,25), d(23,40)\} = |23-40| = 17. \end{aligned}$$

Thus,

$$H(A,B) = \max\{\sup_{a\in A} \delta(a,B), \sup_{b\in B} \delta(A,b)\} = \max\{24,17\} = 24.$$

**Example 2.3.2.**
Consider the metric space $(\mathbb{R}, d)$, where $d$ is the usual metric on $\mathbb{R}$. Consider $A = \{7,19\}$ and $B = \{23,68\}$ are the subsets of $\mathbb{R}$. We find the Hausdorff distance between the set $A$ and the set $B$.
Here the Hausdorff distance between the set $A$ and the set $B$ is defined by:

$$H(A,B) = \max\{\sup_{a\in A} \delta(a,B), \sup_{b\in B} \delta(A,b)\}.$$

Where as the infimum distance for any point $a \in \mathbb{R}$ and a subset $B$ of $\mathbb{R}$ is given by $\delta(a, B) = \inf\{d(a, b) : b \in B\}$ and the supremum distance between two given subsets $A$ and $B$ of $M$ is given by $D(A, B) = \sup\limits_{a\in A}\{\delta(a, B) : b \in B\}$.

Let $a \in A$ and $a = 10$, so

$$\delta(10, B) = \inf\{d(10, 23) : 23 \in B\} = d(10, 23) = |10 - 23| = 13.$$

Now

$$\begin{aligned} D(A, B) = D(\{7, 19\}, \{23, 68\}) &= \sup\{\delta(7, \{23, 68\}), \delta(19, \{23, 68\})\} \\ &= \sup\{d(7, 23), d(19, 23)\} = |7 - 23| = 16. \end{aligned}$$

Again

$$\begin{aligned} D(A, B) = D(\{7, 19\}, \{23, 68\}) &= \sup\{\delta(\{7, 19\}, 23), \delta(\{7, 19\}, 68)\} \\ &= \sup\{d(19, 23), d(19, 68)\} = |19 - 68| = 49. \end{aligned}$$

Thus,

$$H(A, B) = \max\{\sup_{a\in A} \delta(a, B), \sup_{b\in B} \delta(A, b)\} = \max\{16, 49\} = 49.$$

**Example 2.3.3.**

Consider the metric space $(\mathbb{R}, d)$, where $d$ is the usual metric on $\mathbb{R}$. Consider $A = [3, 13]$ and $B = [15, \infty)$ are the subsets of $\mathbb{R}$. We find the Hausdorff distance between the set $A$ and the set $B$. Here the Hausdorff distance between the set $A$ and the set $B$ is defined by:

$$H(A, B) = \max\{\sup_{a\in A} \delta(a, B), \sup_{b\in B} \delta(A, b)\}.$$

Whereas the infimum distance for any point $a \in \mathbb{R}$ and a subset $B$ of $\mathbb{R}$ is given by $\delta(a, B) = \inf\{d(a, b) : b \in B\}$ and the supremum distance between two given subsets $A$ and $B$ of $M$ is given by $D(A, B) = \sup\limits_{a\in A}\{\delta(a, B) : b \in B\}$.

Let $a \in A$ and $a = 12$, so

$$\delta(12, B) = \inf\{d(12, 15) : 15 \in B\} = d(12, 15) = |12 - 15| = 3.$$

Now

$$\begin{aligned} D(A, B) = D([3, 13], [15, \infty)) &= \sup\{\delta([3, 13], 15), \delta([3, 13], \infty)\} \\ &= d(13, \infty) = |13 - \infty| = \infty. \end{aligned}$$

Once again

$$\begin{aligned} D(A, B) = D([3, 13], [15, \infty)) &= \sup\{\delta(3, [15, \infty)), \delta(13, [15, \infty))\} \\ &= d(13, \infty) = |13 - \infty| = \infty. \end{aligned}$$

Thus,

$$H(A, B) = \max\{\sup_{a \in A} \delta(a, B), \sup_{b \in B} \delta(A, b)\} = \max\{\infty, \infty\} = \infty.$$

**Example 2.3.4.**

Consider two nonempty sets $M_1$ and $M_2$ of $M = \mathbb{R}^2$ which are defined respectively as follows:

$$M_1 = \{(m, n)| \ m^2 + n^2 \leq 1 \ \ \wedge \ \ \forall \ m, n \in M\}.$$

and

$$M_2 = \{(m, n)| \ 0 \leq m \leq 3, \ \ 0 \leq n \leq 1 \ \ \wedge \ \ \forall \ m, n \in M\}.$$

Figure **2.1** depicts the sets. Following from Definition 2.3.1, $\delta(a, M_2) = \inf\{d(a, b) : b \in M_2\}$, the set $\delta(a, M_2)$ represents the all distances from every point of $b \in M_2$ to the nearest point $a \in M_1$, where as $(m, n) \in a, b$. It can be observed that if $a \in M_1 \cap M_2$, then it follows that $\delta(a, M_2) = 0$, where as if $b \in M_2 \setminus M_1$, then $\delta(a, M_2)$ can be obtained by using the line from $b$ to the origin which is shown in the following Figure **2.2.** Now we observe the point which produce the largest distance. It is the vertex point which is upper right point i.e. $(3, 1)$ of the rectangle. Therefore, it follows that $D(M_1, M_2)$ is equal to the distance from the point

$\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$ which is on the circle and to the point $(3, 1)$ which is a vertex of the rectangle.

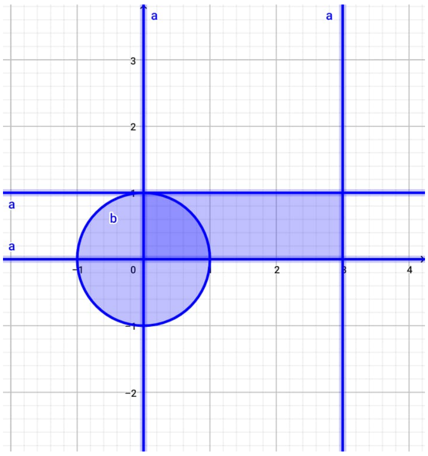


FIGURE 2.1: Geometrical representation of sets $M_1$ and $M_2$.

And the following graph is showing the infimum distance.

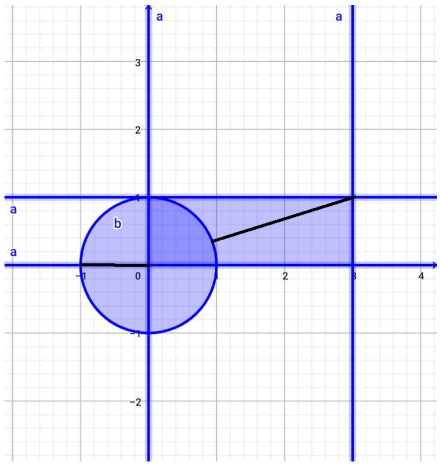


FIGURE 2.2: Infimum distance and the distance between $M_1$ and $M_2$ and also the distance between $M_2$ and $M_1$.

In this Figure the dark shaded area is showing all these points which are representing the infimum distances $\delta(a, M_2) = 0$. Moreover, this Figure is showing that $D(M_1, M_2) = 1$ and $D(M_2, M_1) = \sqrt{10} - 1$. Thus, now we have to actually find the following distance between the points as given below:

$$D(M_1, M_2) = d\left(\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right), (3, 1)\right) = \sqrt{10} - 1,$$

where $D(M_1, M_2) = \sup\{\delta(a, M_2) : a \in M_1\}$, and
$d(a, b) = \sqrt{(m_1 - m_2)^2 + (n_1 - n_2)^2}$, where $(m, n) \in a, b$ and $a = (m_1, n_1)$, $b = (m_2, n_2)$ and also $a \in M_1$ and $b \in M_2$. Now, we have to actually calculate $D(M_1, M_2)$. To calculate that we can take any of the points at the bottom lower of left quadrant on circle with unit radius. Let us take the point $(-1, 0)$ and we find $D(M_2, M_1) = d((-1, 0), (0, 0)) = 1$. Thus, we have the Hausdorff distance is $H(M_1, M_2) = \max\{1, \sqrt{10} - 1\} = \sqrt{10} - 1$.

## 2.4 Continuous Mappings

The concept of continuity of an operator is frequently used in the theory of fixed point. The following concepts are frequently used in the upcoming chapters:

**Definition 2.4.1.**
“Let $M = (M, d_M)$ and $N = (N, d_N)$ be metric spaces. A mapping $T : M \to N$ is said to be continuous at a point $m_0 \in M$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_N(Tm, Tm_0) < \epsilon \quad \text{whenever} \quad d_M(m, m_0) < \delta$$

for all $m$. $T$ is said to be continuous if it is continuous at every point of $M$” [18].

**Theorem 2.4.2.**
“A mapping $T : M \to N$ of a metric space $(M, d_M)$ into a metric space $(N, d_N)$ is continuous at a point $m_0 \in M$ if and only if $m_t \to m_0$ implies $Tm_t \to Tm_0$.

*Proof.*
Assume $T$ to be continuous at the point $m_0$. Then for a given $\epsilon > 0$; there is a $\delta > 0$ such that $d_M(m, m_0) < \delta$ implies $d_N(Tm, Tm_0) < \epsilon$. Let $m_t \to m_0$ and then there is a $t_0 \in \mathbb{N}$ such that for all $t > t_0$ we have $d_M(m_t, m_0) < \delta$. Hence for all $t > t_0$, $d_N(Tm_t, Tm_0) < \epsilon$. By definition, this means that $Tm_t \to Tm_0$. Conversely, we assume that $m_t \to m_0$ implies $Tm_t \to Tm_0$, and prove that then $T$ is continuous at a point $m_0$. Suppose this is false, then there is an $\epsilon > 0$ such that for every $\delta > 0$ there is a $m \neq m_0$ satisfying $d_M(m, m_0) < \delta$ but

$d_N(Tm, Tm_0) \geq \epsilon$. In particular, for $\delta = \dfrac{1}{t}$ there is a sequence $m_t$ satisfying: $d_M(m_t, m_0) < \dfrac{1}{t}$ but $d_N(Tm_t, Tm_0) \geq \epsilon$. Clearly $m_t \to m_0$ but $Tm_t$ does not converge to $Tm_0$. This contradicts $Tm_t \to Tm_0$ and proves the theorem" [18]. □

**Definition 2.4.3.**

"Let $T : D \to \mathbb{R}$ and let $m_0 \in D$. We say that $T$ is lower semi-continuous (L.S.C.) at $m_0$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $T(m_0) - \epsilon \leq T(m)$ for all, $m \in B(m_0, \delta) \cap D$.

Similarly, we say that $T$ is upper semi-continuous (U.S.C.) at $m_0$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $T(m) \leq T(m_0) + \epsilon$ for all, $m \in B(m_0, \delta) \cap D$.

It is clear that $T$ is continuous at $m_0$ if and only if $T$ is lower semi-continuous and upper semi-continuous at this point" [107].

It is interesting to mention here that a mapping may be lower or upper semi-continuous whether it is left or right continuous or not continuous. For further detail see [13–20, 107]. The following examples illustrate the above concepts:

**Example 2.4.1.**

Let $T\colon \mathbb{M} \to \mathbb{M}$ be a function, where $M = \mathbb{R}$ is defined in the way as given below:

$$T(m) = \begin{cases} \dfrac{1}{m}; & \text{if, } m < 0, \\ 0; & \text{if, } m = 0, \\ -\dfrac{1}{m}; & \text{if, } m > 0. \end{cases}$$

Then this mapping is upper semi-continuous and if $m_0 \to 0$ then $T(m_0) \to -\infty < 0 = T(0)$.

The right and left limit of the mapping is $-\infty$ which is different from the value of the mapping that is 0.

**Example 2.4.2.**

Let $T\colon \mathbb{M} \to \mathbb{M}$ be a function, where $M = \mathbb{R}$ is defined in the way as given below:

$$T(m) = \begin{cases} m^2; & \text{if, } m \neq 0, \\ -1; & \text{if, } m = 0, \end{cases}$$

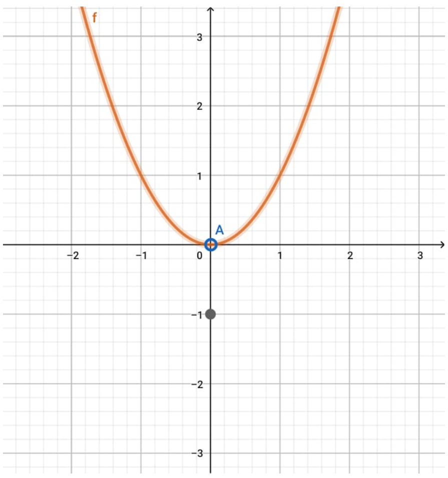


FIGURE 2.3: This graph depicts that $m = 0$ is a lower semi-continuous point.

The function is lower semi-continuous at the point $m = 0$.

**Example 2.4.3.**

Let $T\colon \mathbb{M} \to \mathbb{M}$ be a function, where $M = \mathbb{R}$ is defined in the way as given below:

$$T(m) = \begin{cases} -m^2; & \text{if, } m \neq 0, \\ 1; & \text{if, } m = 0, \end{cases}$$

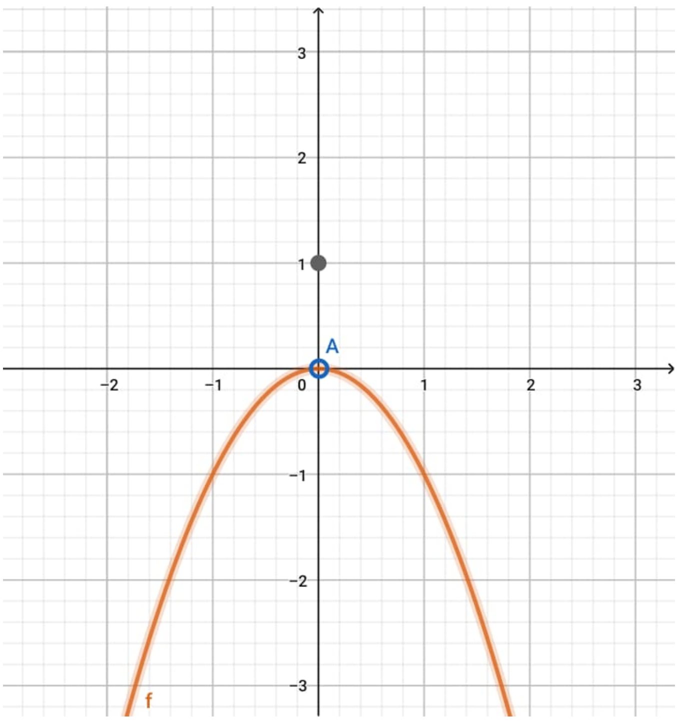


FIGURE 2.4: This graph depicts that $m = 0$ is an upper semi-continuous point.

The function is upper semi-continuous at $m = 0$.

This concept can be further clarified with the help of following functions and their graphs: Let $T\colon \mathbb{R} \to \mathbb{R}$ be a function defined in the following way:

$$T(m) = \begin{cases} m; & \text{if, } m < 1, \\ m-1; & \text{if, } 1 \leq m \leq 2, \\ -m+3; & \text{if, } 2 \leq m < 3, \\ 1; & \text{if, } m = 3, \\ m-3; & \text{if, } 3 < m \end{cases}$$

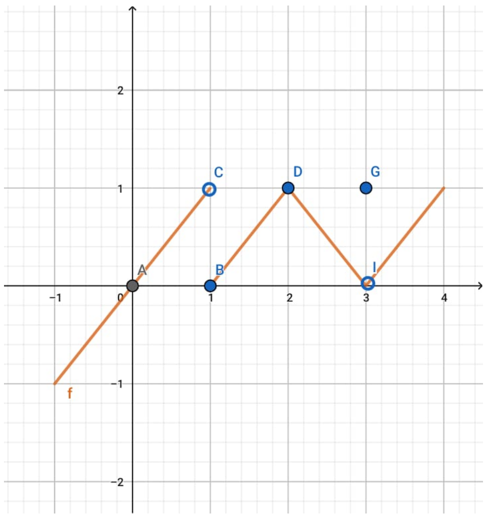


FIGURE 2.5: This graph depicts that $m = 1$ is a lower semi-continuous point.

The function is a lower semi-continuous at $m = 1$.

Now consider the following mapping:

Let $T\colon \mathbb{R} \to \mathbb{R}$ be a function defined in the following way:

$$T(m) = \begin{cases} m; & \text{if, } -1 \leq m \leq 0, \\ -2m^2; & \text{if, } 0 \leq m < 1, \\ \sqrt{-m^2+4m-3}; & \text{if, } 1 \leq m < 2, \\ 0; & \text{if, } m = 1, \\ \sqrt{-m^2+4m-3}; & \text{if, } 2 < m \leq 3 \end{cases}$$

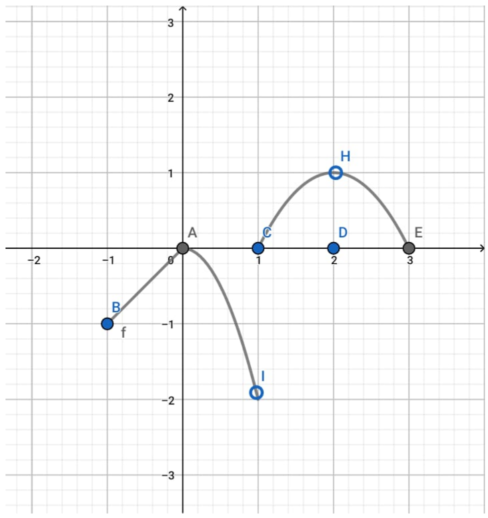


FIGURE 2.6: This graph depicts that $m = 1$ is an upper semi-continuous point.

This function is an u. s. continuous at $m = 1$.

To define the concept of $T$-orbitally continuous maps, the following concept is required:

**Definition 2.4.4.**

"Let $T\colon M \to M$ and for some $m_0 \in M$,

$$O_T(m_0) = \{m_0, Tm_0, T^2m_0, ....\}$$

be the orbit of $m_0$" [70].

**Example 2.4.4.**

Consider a set $M = [0, 2]$ with usual metric. Define $T\colon M \to M$ as:

$$T(m) = \begin{cases} \dfrac{m}{2}; & \text{if } \ m \in [0, 1), \\ \dfrac{1+m}{2}; & \text{if } \ m \in [1, 2). \end{cases}$$

Assuming $m_0 = \dfrac{1}{4} \in M$ then we have

$$\begin{aligned} O_T(m_0) &= \{m_0, Tm_0, T^2m_0, ....\} \\ &= \left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ....\right\} \\ &= \left\{\frac{1}{2^t}, \ \ t \geq 2, \ \ t \in \mathbb{Z}^+\right\}. \end{aligned}$$

Let $m_0 = \dfrac{3}{2} \in M$ then we have

$$\begin{aligned} O_T(m_0) &= \{m_0, Tm_0, T^2m_0, ....\} \\ &= \left\{\frac{3}{2}, \frac{5}{4}, \frac{9}{8}, ....\right\}. \end{aligned}$$

**Example 2.4.5.**

Consider a set $M = [-2, 2] \times [-2, 2]$ and define a self mapping $T\colon M \to M$ on the set $M$ in the following way:

$$T(m) = T(m_1, m_2) = \begin{cases} \left(\dfrac{m_1}{2}, \dfrac{m_2}{2}\right); & \text{if } m_1, m_2 \geq 0, \\ (2, 0); & \text{otherwise.} \end{cases}$$

Obviously, the mapping $T$ is not continuous at $(0, 0) \in M$.

Assuming $m = (m_1, m_2) \in M$ such that $0 < m_1, m_2 < 1$, so we have $O_T(m) = \left\{m, \dfrac{m}{2}, \dfrac{m}{4}, ....\right\}$.

**Definition 2.4.5.**

"A function $T$ from a nonempty set $M$ into the set of real numbers *i. e.* $T\colon M \to \mathbb{R}$ is said to be $T$-orbitally lower semi continuous at $v \in M$ if for a sequence $m_t \subset O_T(m)$ and $m_t \to v$, implies

$$T(v) \leq \lim_{t\to\infty} \inf T(m_t)\text{"} \quad [70].$$

**Example 2.4.6.**

Consider a set $M = [-2, 2]$ and a self map $T\colon M \to M$ defined as:

$$T(m) = \frac{m}{2}.$$

For $m_0 \in (0, 2)$, the orbit of $m_0$ with respect to $T$ is given by

$$O_T(m_0) = \left\{m_0, \frac{m_0}{2}, \frac{m_0}{4}, ....\right\}.$$

Let $\{m_t\}$ be any sequence in $O_T(m_0)$ which converges to zero. Now consider a function $S\colon M \to \mathbb{R}$ defined by $S(m) = |m|$.

One can easily check that $\liminf_{t\to\infty} S(m_t) = 0 = S(0)$.

Hence, $S$ is $T$-orbitally lower semi-continuous at $m = 0$.

## 2.5 Fixed Point

**Definition 2.5.1.**

An element $m \in M$ of any nonempty set $M$, is said to be a fixed point for a self mapping $T : M \to M$ if $Tm = m$. The set of all the fixed points is represented by $FixT$, *i. e.*

$$FixT = \{m \in M : Tm = m\}.$$

The following graphs depict the concept of fixed points.

**Example 2.5.1.**

1. Let $M = \mathbb{R}$. A self mapping $T : M \to M$ such that $T(m) = m^3$ has three fixed points $i.e, m = -1, 0, 1$.

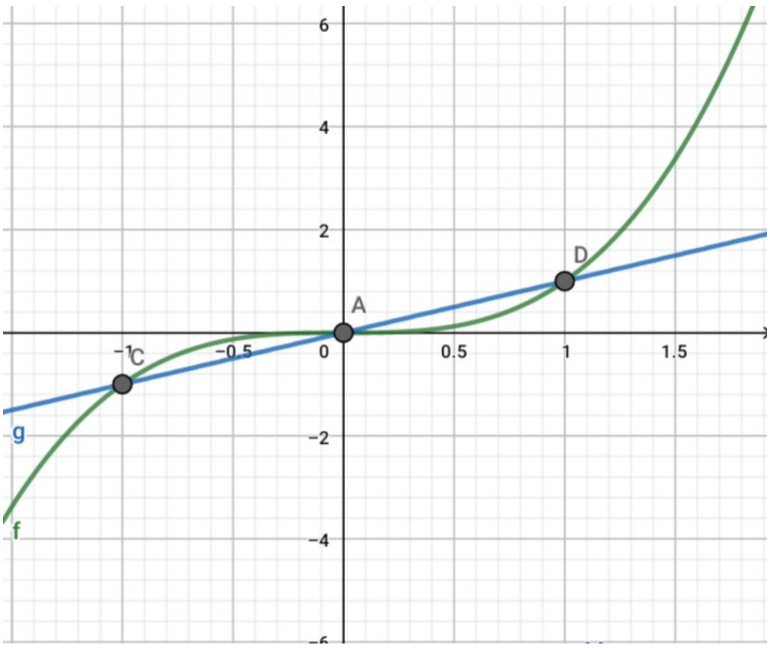


FIGURE 2.7: The graph of the mapping defined by $T(m) = m^3$ depicting that the mapping having three fixed points.

2. Let $M = C\left[0, \frac{1}{2}\right]$. A self mapping $T : M \to M$ such that
$$T(m(t)) = t\,(m(t) + 1)$$
has a unique fixed point $m^*(t) = \frac{t}{1-t}$.

3. Let $M = \mathbb{R}^2$. A self mapping $T \colon M \to M$ such that
$$T(m_1, m_2) = \left(\frac{m_1}{a} + b, \frac{m_2}{c} + b\right),$$
where $(m_1, m_2) \in M$ and $a, c > 1$. Then $\left(\frac{ab}{a-1}, \frac{cb}{c-1}\right)$ is a fixed point for the given self mapping $T$.

4. A translation mapping has no fixed point.

5. $T(m) = 2^m$ and $T(m) = \log_2 m$ has no fixed point.

6. Let $M = \mathbb{R}$. A self mapping $T : M \to M$ such that $T(m) = m^2 + m + 1$ has no fixed point.

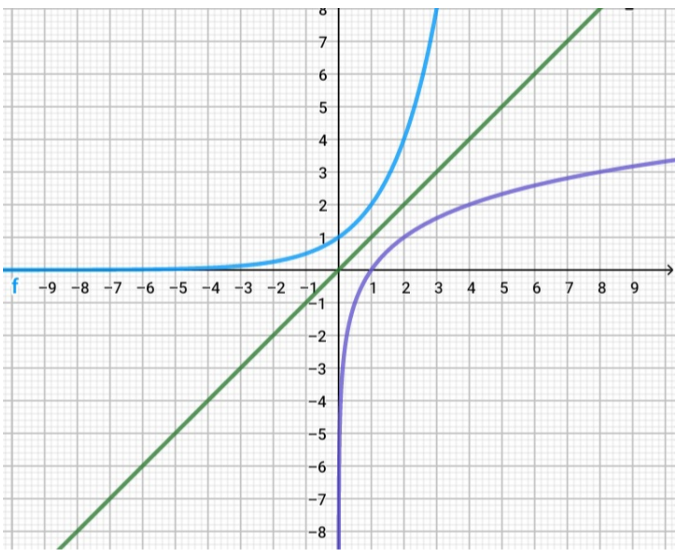

FIGURE 2.8: The graph of the functions $T(m) = 2^m$ and $T(m) = \log_2 m$ depicting that the functions having no fixed point.

It is worth to mention here that the fixed point of any real valued function $y = T(m)$ is in fact the point of intersection of the function $T(m)$ and line $y = m$.

**Definition 2.5.2.**

"A mapping $T\colon M \to M$ is called a Picard operator if $T$ has a unique fixed point $m_0 \in M$ such that $\lim\limits_{n\to 0} T^n m = m_0$ for all $m \in M$" [108].

**Example 2.5.2.**

Let $M = [0, 1]$. Define a map $T\colon M \to M$ as follows:

$$Tm = \frac{m}{2},$$

then it is simple to show that the mapping $T$ is a Picard operator and simply it has a unique fixed point $m_0 = 0$ which can be verified easily.

In a multivalued function every input is assigned to several values i. e. outputs that is similar to a function which is also known as multi-function or set-valued function or many-valued function. Moreover, it is a mapping from a set $M$ to a set $N$ that associates every $m \in M$ to more than one value $n \in N$. To achieve the goals of this research, the following definition is necessary.

**Definition 2.5.3.**

"A mapping $T : M \to P(M)$ is said to be a multivalued, if for each element $m \in M$, $Tm$ is a nonempty subset of $M$. In other words, a multivalued map $T$ from a set $M$ to $P(M)$ is a nonempty subset of the product set of $M \times P(M)$. That is, if $Tm \subset P(M)$ is a nonempty set, then $T$ is said to be a multivalued map and the image of an element $m \in M$ under $T$ is denoted by $Tm$ and defined by:

$$Tm = \{n \in P(M) : (m, n) \in Tm\} \subset P(M),$$

where $M$ and $P(M)$ are nonempty sets.

Notice that integer power, hyperbolic, exponential and trigonometric functions are all single valued but their inverses are the examples of multivalued mapping" [109].

One can easily observe that these mappings are not functions being one-to-one or one-to-many correspondence. A multivalued mapping or a set valued mapping is in fact a total relation; that means; every input maps on one or more outputs. Where as, a function is "well-defined" that associates one, and only one, output to any particular input. The most simple example of multivalued mapping is $T : [0, \infty) \to P(\mathbb{R})$ defined as below:

$$T(c) = \{\pm\sqrt{c}\} = \{\sqrt{c}, -\sqrt{c}\}, \quad \text{for all,} \quad c \in \mathbb{R}^+.$$

**Example 2.5.3.**

Consider $M = [0, 1]$ and $N(M) = \{A \subset M : A \neq \emptyset\}$. Define a map $T : M \to N(M)$ as

$$Tm = [0, m]$$

is a multivalued mapping. Its graph is as given under:

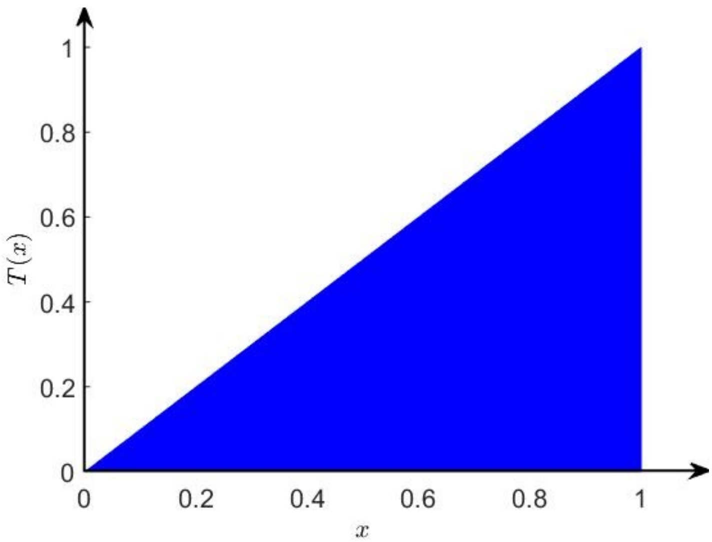


FIGURE 2.9: This graph is depicting multivalued mapping.

**Example 2.5.4.**

Consider $M = [0, 1]$ and $CB(M) = \{A \subset M : A \neq \emptyset\}$. Now define $T : M \to CB(M)$ as

$$Tm = \begin{cases} \Big[0, 1\Big]; & \text{if} \quad m \neq \dfrac{1}{2}, \\ \\ \left[\dfrac{1}{2}, 1\right]; & \text{if} \quad m = \dfrac{1}{2}, \end{cases}$$

is a multivalued mapping.

The graph of this mapping is shown in the following figure.

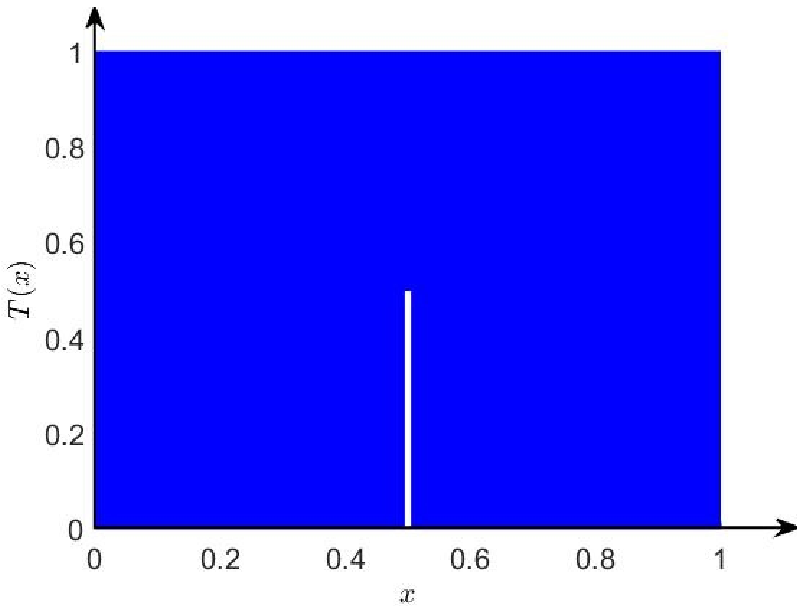


FIGURE 2.10: This graph is depicting multivalued mapping.

**Definition 2.5.4.**

"An element $m \in M$ for any nonempty set $M$, is said to be a fixed point for a mapping $T : M \to P(M)$ if $m \in Tm$" [109].

Here $Tm$ is nonempty subset of $M$.

**Example 2.5.5.**

Let $M = [0, 1]$. Define a mapping $T : M \to P(M)$ by

$$Tm = [0, m^2],$$

then 1 and 0 are fixed points of the mapping $T$.

**Definition 2.5.5.**

"Let $T_1, T_2 : M \to P(M)$ be two multivalued mappings. Then a point $m_0 \in M$ is said to be a common fixed point for the mappings $T_1$ and $T_2$ if

$$m_0 \in T_1 m_0 \cap T_2 m_0\text{" [109].}$$

**Example 2.5.6.**

Consider $M = [0, 1]$ and let $m \in M$. Now define $T_1, T_2 : M \to P(M)$ as

$$T_1 m = \left[0, \frac{m}{4}\right]; \qquad \forall \qquad m \in M$$

and

$$T_2 m = \left[0, \frac{m}{2}\right]; \qquad \forall \qquad m \in M$$

then it is simple to find that 0 is a common fixed point for mappings $T_1$ and $T_2$.

**Example 2.5.7.**

Consider $M = [0, 2]$ and assume that $m, n \in M$ such that $n > m$. Now define $T_1, T_2 : [m, n] \to P([m, n])$ as

$$T_1 a = \begin{cases} \{m\}; \quad \text{if} \quad a \in \{m, n\}; \\ \\ [a, n]; \quad \text{if} \quad m < a < n, \end{cases}$$

and

$$T_2 a = [m, a]; \quad \forall; \quad a \in [m, n]$$

then for each point $a \in [m, n]$ is a common fixed point for mappings $T_1$ and $T_2$.

## 2.6 Some Abstract Spaces

In this section the concepts of $b$-metric space, $S$-metric space, $S_b$-metric space and uniform space are explicated.

### 2.6.1 $b$-Metric Space

Bakhtin [53] and Czerwick [33, 34, 37] developed the idea of $b$-metric space. In literature a lot of consequences of this study can be found. See for examples [64–67, 69, 70].

**Definition 2.6.1.**

"Let $M$ be a nonempty set and $b \geq 1$ be a given real number. A function $d : M \times M \to [0, \infty)$ is called $b$-metric if it satisfies the following properties for each $m_1, m_2, m_3 \in M$:

**(B1)** $d(m_1, m_2) = 0 \Leftrightarrow m_1 = m_2$,

**(B2)** $d(m_2, m_1) = d(m_1, m_2)$

**(B3)** $d(m_3, m_1) \leq b[d(m_3, m_2) + d(m_2, m_1)]$.

The pair $(M, d)$ is called a $b$-metric space" [70].

It is noticed that the $b$-metric class is bigger than the class of metric space. Moreover, it is simple to observe that every $b$-metric space is a metric space when $b = 1$, but we can find examples of such $b$-metrics which are not metrics.

**Example 2.6.1.**

Consider a nonempty set as defined below:

$$M = \ell_p(\mathbb{R}) = \left\{ \{m_t\} \subset \mathbb{R} : \sum_{t=1}^{\infty} |m_t|^p < \infty \right\}; \quad \text{for} \quad 0 < p < 1.$$

Let us define a mapping $d : M \times M \to \mathbb{R}^+$ as follows:

$$d(m, n) = \left( \sum_{t=1}^{\infty} |m_t - n_t|^p \right)^{\frac{1}{p}} \quad \text{for} \quad m = \{m_t\}, \quad n = \{n_t\}.$$

It is simpe to verify that $(M, d)$ is a $b$-metric space by proving all the properties with coefficient $b = 2^{\frac{1}{p}}$ [70].

**Example 2.6.2.**

Let $M = \{0, 1, 2\}$. A mapping $d_b \colon M \times M \to [0, +\infty)$ is defined as:

$$d_b(0,0) = d_b(1,1) = d_b(2,2) = 0,$$

$$d_b(0,1) = d_b(1,0) = 1$$

$$d_b(1,2) = d_b(2,1) = 1,$$

$$d_b(0,2) = d_b(2,0) = m \geq 2,$$

then it can be easily verified that $d_b$ is a $b$-metric for $b = \dfrac{m}{2} \geq 1$, but it is not a metric for $m > 2$.

**Example 2.6.3.**

Let $(M, d)$ be a metric space. Define a function

$$d_1(a, b) = [d(a, b)]^m,$$

for a real number $m > 1$, then $d_1$ is a $b$-metric with $b = 2^{m-1}$.

The following example depicts that every $b$-metric space need not be continuous.

**Example 2.6.4.**

Let $M = \mathbb{N} \cup \{0\}$ and let $d_b \colon M \times M \to [0, +\infty)$ is defined by

$$d_b(m, n) = \begin{cases} 0 & \text{if } m = n, \\ \left|\frac{1}{m} - \frac{1}{n}\right| & \text{if one of } m, n \text{ is even and the other is even or } \infty. \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

It can be checked that for all $m, n, p \in M$, we have

$$d_b(m, p) \leq \frac{5}{2}\left[d_b(m, n) + d_b(n, p)\right].$$

Thus $(M, d_b)$ is a $b$-metric space with $b = \dfrac{5}{2}$.

Choose a sequence $m_t = 2t$ for each $t \in \mathbb{N}$, with $\lim\limits_{t\to\infty} m_t = \infty$,
but

$$\lim_{t\to\infty} d_b(m_t, 1) = 2 \nrightarrow 5 = d_b(\infty, 1) \ \text{ as } \ t \to \infty.$$

**Definition 2.6.2.**

"Let $(M, d_b)$ be a $b$-metric space and $\{m_t\}$ be a sequence in $M$ then:

- $\{m_t\}$ is convergent if and only if there exists $m \in M$ such that $d(m_t, m) \to 0$ as $t \to \infty$ and we write $\lim_{t \to \infty} m_t = m$.
- $\{m_t\}$ is Cauchy if and only if $d(m_t, m_s) \to 0;\;$ as $\;t, s \to \infty$.
- The $b$-metric space $(M, d_b)$ is called complete if every Cauchy sequence is convergent in it" [70].

### 2.6.2 $S$-Metric Space

This section is devoted to $S$-metric space and related concepts. The notion of $S$-metric space was given by Sedghi [110] which is further utilized by Prudhvi and Mlaiki [111, 112].

**Definition 2.6.3.**
"Let $M$ be a nonempty set. An $S$-metric on $M$ is a function $S : M^3 \to [0, \infty)$ that satisfies the following conditions, for all $m, n, c, t \in M$:

$(S1.)$ $S(m, n, c) = 0 \Leftrightarrow m = n = c$,

$(S2.)$ $S(m, m, n) = S(n, n, m)$,

$(S3.)$ $S(m, n, c) \leq S(m, m, t) + S(n, n, t) + S(c, c, t)$,

then the pair $(M, S)$ is called an $S$-metric space" [110].

**Example 2.6.5.**
Let $M = \mathbb{R}$ be a set of reals and $S : M^3 \to [0, \infty)$ be a function defined as:

$$S(m, n, c) = |m - c| + |n - c|,$$

then it is very simple to verify that $(M, S)$ is an $S$-metric space [110].

**Example 2.6.6.**
Let $M = \mathbb{R}^2$ and $S : M^3 \to [0, \infty)$ be a function defined as:

$$S(m, n, c) = d(m, n) + d(n, c) + d(c, m), \quad \text{for all} \quad m, n, c \in \mathbb{R}^2,$$

where $d$ is euclidean metric on $\mathbb{R}^2$, then it is easy to verify that $(M, S)$ is an $S$-metric space.

To prove the fixed point results on $S$-metric spaces Sedghi [110] used the following concepts:

**Definition 2.6.4.**
"Let $(M, S)$ be an $S$-metric space.

- A sequence $\{m_t\}$ in $M$ converges to $m$ if and only if $S(m_t, m_t, m) \to 0$ as $t \to \infty$. That is for each $\epsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that for all $t > t_0$, $S(m_t, m_t, m) < \epsilon$ and we denote this by $\lim_{t\to\infty} m_t = m$.
- A sequence $\{m_t\}$ in $M$ is called a Cauchy sequence if for each $\epsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that $S(m_t, m_t, m_s) < \epsilon$ for each $t, s > t_0$.
- The $S$-metric space $(M, S)$ is said to be complete if every Cauchy sequence is convergent" [110].

### 2.6.3 $S_b$-Metric Space

Sedghi [110] introduced the idea of an $S$-metric space and produced some fixed point results in this settings.

**Definition 2.6.5.**
"Let $M$ be a nonempty set and let $b \geq 1$ be a given real number. A function $S_b : M^3 \to [0, \infty)$ is said to be $S_b$-metric if and only if for all $m_1, m_2, m_3, t \in M$: the following conditions hold:

$(S_b1.)$ $S_b(m_1, m_2, m_3) = 0 \Leftrightarrow m_1 = m_2 = m_3$,

$(S_b2.)$ $S_b(m_1, m_1, m_2) = S_b(m_2, m_2, m_1)$,

$(S_b3.)$ $S_b(m_1, m_2, m_3) \leq b[S_b(m_1, m_1, t) + S_b(m_2, m_2, t) + S_b(m_3, m_3, t)]$

then the pair $(M, S_b)$ is called a $S_b$-metric space" [101].

**Example 2.6.7.**

Consider a nonempty set $M$ with $card(M) \geq 5$ also assume that $M = M_1 \cup M_2$ is partition of $M$ with $card(M_1) \geq 4$. Let $b \geq 1$ and for all $m_1, m_2, m_3 \in M$, we define

$$S_b(m_1, m_2, m_3) = \begin{cases} 0 & \text{for} \quad m_1 = m_2 = m_3 = 0; \\ 3b & \text{for} \quad (m_1, m_2, m_3) \in M_1^3; \\ 1 & \text{for} \quad (m_1, m_2, m_3) \notin M_1^3, \end{cases}$$

then it can be easily verified that the pair $(M, S_b)$ is an $S_b$-metric space with coefficient $b \geq 1$ [101].

**Example 2.6.8.**

Let $M = \mathbb{R}$ be a set of real numbers and $S_b : M^3 \to [0, \infty)$ be a function with the metric defined as:

$$S_b(m, n, c) = |m - c| + |n - c|,$$

then it is easy to verify that $(M, S_b)$ is an $S_b$-metric space with $b \geq 1$ [110].

The following are some important definitions related to the concepts of $S_b$-metric space:

**Definition 2.6.6.**

"Let$(M, S_b)$ be an $S_b$-metric space and $m_t$ be a sequence in $M$, then

- a sequence $\{m_t\}$ is called convergent if and only if there exists $m \in M$ such that $S_b(m_t, m_t, m) \to 0$, as $t \to \infty$. In this case we write $\lim\limits_{t\to\infty} m_t = m$.
- a sequence $\{m_t\}$ is called a Cauchy sequence if and only if $S_b(m_t, m_t, m_s) \to 0$, as $t, s \to \infty$.
- $(M, S_b)$ is said to be complete if every Cauchy sequence $\{m_t\}$ converges to a point $m \in M$ such that $\lim\limits_{t,s\to\infty} S_b(m_t, m_t, m_s) = \lim\limits_{s\to\infty} S_b(m_s, m_s, m) = m$" [101].

### 2.6.4 Uniform Space

This section adresses the concept of uniform space and its related ideas. A uniform space is a structure on a nonempty set $M$ that was presented by Weil (1937) in terms of subset of $M \times M$. Later on, Tukey (1940) provided an alternate description by using covers of $M$. Some results in this setting are provided in [12, 48, 93, 113]. For the better and detail understanding of uniform spaces consider the Kelley's book [114] and see (e.g. [6, 11]). Here Weil's approach of uniform space structure is taken into consideration.

Now, we recollect certain fundamental concepts and basic definitions which are required subsequently.

**Definition 2.6.7.**
"A subset $\mathcal{U}$ of subsets of $M \times M$ is called a uniformity (or a uniform structure) on $M$ if:

**(i)** $\Delta = \Delta(M) = \{(m, m) : m \in M\} \subseteq G$ for all $G \in \mathcal{U}$;

**(ii)** If $G \in \mathcal{U}$ and $H \subseteq M \times M$ with $G \subseteq H$, then $H \in \mathcal{U}$;

**(iii)** If $G \in \mathcal{U}$, there exists some $H \in \mathcal{U}$ such that, $H^2 \subseteq G$;

**(iv)** If $G, H \in \mathcal{U}$, there exists some $C \in \mathcal{U}$ such that, $C \subseteq G \cap H$;

**(v)** $G \in \mathcal{U}$, implies that $G^{-1} = \{(n, m) : (m, n) \in G\} \in \mathcal{U}$, i. e. each $G \in H$ is symmetric.

In this case, the pair $(M, \mathcal{U})$ is called a uniform space. The members of $\mathcal{U}$ are called vicinities of $\mathcal{U}$. The pair $(M, \mathcal{U})$ without the property **(v)** is called a quasi-uniform space" [114].

**Definition 2.6.8.**
"For a subset $V \in \mathcal{U}$ a pair of points $m$ and $n$ are said to be $V$-close, if $(m, n) \in V$ and $(n, m) \in V$.

Moreover, a sequence $\{m_t\}$ in $M$ is called a Cauchy sequence for $\mathcal{U}$, if for any

$V \in \mathcal{U}$, there exists a $t_0 \geq 1$ such that $m_s$ and $m_t$ are $V$-close for $t, s \geq t_0$.
For $(M,\mathcal{U})$, there is a unique topology $\tau(\mathcal{U})$ on $M$ generated by
$V(m) = \{n \in M | (m,n) \in V\}$, where $V \in \mathcal{U}$" [48].

The followings are very important definitions related to the notion of uniform space:

**Definition 2.6.9.**
"A sequence $\{m_t\}$ in $M$ is convergent to $m$ for $\mathcal{U}$, that is $\lim\limits_{t\to\infty} m_t = m$, if for any $V \in \mathcal{U}$, there exists $t_0 \in \mathbb{N}$ such that $m_t \in V(m)$ for every $t \geq t_0$.
Let $\Delta(M) = \{(m,m) : m \in M\}$ be the diagonal of $M$. For $V \subseteq M \times M$, we define $V^{-1} = \{(m,n)|(n,m) \in V\}$" [48].

**Definition 2.6.10.**
"A uniform space $(M,\mathcal{U})$ is called Hausdorff if the intersection of all the $V \in \mathcal{U}$ is equal to $\Delta$ of $M$, that is, if $(m,n) \in V$ for all $V \in \mathcal{U}$ implies $m = n$" [48].

**Definition 2.6.11.**
"If $V = V^{-1}$ then we say that a subset $V \in \mathcal{U}$ is symmetrical" [48].

To see the further detail it is reffered to see [6, 11, 23, 48, 93, 113, 115–118].
For further learning, now we intend to revisit the concepts of $A$-distance and $E$-distance.

**Definition 2.6.12.**
"Let $(M,\mathcal{U})$ be a uniform space. A function $p : M \times M \longrightarrow [0,\infty)$ is said to be an $A$-distance if for any $V \in \mathcal{U}$, there exists $\delta > 0$ such that if $p(z,m) \leq \delta$ and $p(z,n) \leq \delta$ for some $z \in M$, then $(m,n) \in \mathcal{U}$" [48].

**Definition 2.6.13.**
"Let $(M,\mathcal{U})$ be a uniform space. A function $p : M \times M \longrightarrow [0,\infty)$ is said to be an $E$-distance if

**(i)** $p$ is an $A$-distance,

**(ii)** $p(m,n) \leq p(m,z) + p(z,n)$, $\forall\, m,n,z \in M$" [48].

**Example 2.6.9.**

Consider a uniform space $(M,\mathcal{U})$ and with a metric $d$ on $M$. It is easy to see that $(M,\mathcal{U}_d)$ is a uniform space. Now consider a set having all subsets of $M \times M$ with a "band" $U_\epsilon = \{(m,n) \in M^2 : d(m,n) < \epsilon\}$ for some $\epsilon > 0$ is represented by $\mathcal{U}_d$. Further, for $\mathcal{U} \subseteq \mathcal{U}_d$, the $d$ is an $E$-distance on $(M,\mathcal{U})$ [48].

**Lemma 2.6.14.**

"Let $(M,\mathcal{U})$ be a Hausdorff uniform space and $p$ be an $A$-distance on $M$. Let $\{m_t\}$ and $\{n_t\}$ be sequences in $M$ and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0. Then, for $m, n, z \in M$, the following results hold:

**(a)** If $p(m_t, n) \leq \alpha_n$ and $p(m_t, z) \leq \beta_n$ for all $t \in \mathbb{N}$, then $n = z$.

In particular, if $p(m,n) = 0$ and $p(m,z) = 0$, then $n = z$.

**(b)** If $p(m_t, n_t) \leq \alpha_n$ and $p(m_t, z) \leq \beta_n$ for all $t \in \mathbb{N}$, then $\{n_t\}$ converges to $z$.

**(c)** If $p(m_t, m_s) \leq \alpha_n$ for all $t, s \in \mathbb{N}$ with $s > t$, then $\{m_t\}$ is a Cauchy sequence in $(M,\mathcal{U})$" [48].

**Definition 2.6.15.**

"Let $p$ be an $A$-distance. A sequence in a uniform space $(M,\mathcal{U})$ with an $A$-distance is said to be a $p$-Cauchy if for every $\epsilon > 0$, there exists $t_0 \in \mathbb{N}$ such that $p(m_t, m_s) < \epsilon$ for all $t, s \geq t_0$" [48].

**Definition 2.6.16.**

"Let $(M,\mathcal{U})$ be a uniform space and $p$ be an $A$-distance on $M$. Then:

**(i)** $M$ is $S$-complete for if each $p$-Cauchy sequence $\{m_t\}$, there exists $m \in M$, $\lim_{t\to\infty} p(m_t, m) = 0$.

**(ii)** $M$ is $p$-Cauchy complete if for each $p$-Cauchy sequence $\{m_t\}$, there is $m$ in $M$ with $\lim_{t\to\infty} m_t = m$ with respect to $\tau(\mathcal{U})$.

**(iii)** $T : M \to M$ is $p$-continuous if $\lim_{t\to\infty} p(m_t, m) = 0$ then we have

$$\lim_{t\to\infty} p(Tm_t, Tm) = 0\text{"} \quad [48].$$

**Remark 2.6.17.**

"Let $(M,\mathcal{U})$ be a Hausdorff uniform space which is $S$-complete, then it is also $p$-Cauchy complete" [48].

## 2.7 Banach Contraction Principle (BCP) and its Extensions

This section provides the most fascinating result of fixed point theory given by Banach in 1922 and it also contains some generalizations of BCP. It is quite impossible to observe all extensions of BCP since there is enormous literature about it, however an attempt is made to provide some generalizations which are obtained under mild modified conditions. See for examples [1, 23, 57–59, 62, 63, 72, 116, 119–131].

**Theorem 2.7.1.**

"Let $(M,d)$ be a complete metric space and let $T\colon M \to M$ be a contraction mapping. Then $T$ has unique fixed point $m_0$, and for each $m \in M$,

$$\lim_{n\to\infty}\{T^n m\} = m_0.$$

Moreover,

$$d(T^n m, m_0) \leq \frac{k^n}{1-k} d(m, Tm)\text{"} \quad [17].$$

The validity and reliability of this contraction principle is lying in the fact that it provides a contraction algorithm to converge at a fixed point along with error estimates.

Many authors have extended Banach contraction principle by using different contractive conditions. Some of the such results are stated below:
The very first generalization of BCP is given by Edelstein in which he consider compact space instead of complete space.

**Theorem 2.7.2.**

"Let $M$ be a metric space and $A$ be a contractive mapping of $M$ into itself such that there exists a point $m \in M$ whose sequence of iterates $\{A^t m\}$ contains a convergent sub sequence $\{A^{ti} m\}$; then

$$\xi = \lim_{i \to \infty} A^{ti} m \in M$$

is a unique fixed point" [132].

The next theorem is given by Rakotch in which he used a monotonic decreasing function in the place of a constant.

**Theorem 2.7.3.**

" If $A$ is a contractive mapping of a metric space $M$ into itself, and there exists a subset $N \subset M$ and a point $m_0 \in N$ such that

$$p(m, m_0) - p(Am, Am_0) \geq 2p(m_0, Am_0),$$

for every $m \in M - N$ and $A$ maps $N$ into a compact subset of $M$, then there exists a unique fixed point" [82].

The result given below is provided by Chatterjea in which a new type of contraction

is applied for giving a new result.

**Theorem 2.7.4.**

"Let $(M, d)$ be a complete metric space. A self mapping $T : M \to M$ is a contractive mapping if there exists $k \in [0, \frac{1}{2})$ such that

$$d(Tn, Tm) \leq k(d(m, Tn) + d(n, Tm)), \ \forall \ m, n \in M.$$

Then $T$ has a unique fixed point" [29].

The following extension is based on a new contractive condition in proving fixed

point that is used by Bianchini.

**Theorem 2.7.5.**

"Let $(M, d)$ be a complete metric space. A self-mapping $T : M \to M$ is a contractive mapping if there exists $k \in [0, 1)$ such that

$$d(Tn, Tm) \leq k \max\{d(n, Tn), d(m, Tm)\}, \ \forall \ m, n \in M.$$

Then $T$ has a unique fixed point" [32].

The next provided theorem is given by Kannan in which a new contractive

condition is used for establishing a certain generalized result.

**Theorem 2.7.6.**

"Let $(M, d)$ be a complete metric space. A self-mapping $T : M \to M$ is a contractive mapping if there exists $k \in [0, \frac{1}{2})$ such that

$$d(Tn, Tm) \leq k(d(n, Tn) + d(m, Tm)), \ \forall \ m, n \in M.$$

Then $T$ has a unique fixed point" [30].

The next extension is based on a new defined contractive condition in detecting fixed point that is used by Reich.

**Theorem 2.7.7.**

"Let $(M, d)$ be a complete metric space. A self-mapping $T : M \to M$ is a contractive mapping if there exist non-negative real numbers $k_1, k_2, k_3$, satisfying $k_1 + k_2 + k_3 < 1$ such that

$$d(Tm, Tn) \leq k_1 d(m, Tm) + k_2 d(n, Tn) + k_3 d(m, n), \ \forall \ m, n \in M.$$

Then $T$ has a unique fixed point" [27].

The theorem given below is provided by Hardy in which a new type of contractive mapping is defined for proving a new result.

**Theorem 2.7.8.**

"Let $(M, d)$ be a complete metric space. A self-mapping $T : M \to M$ is a contractive mapping if there exist non-negative real numbers $a_0, b_0, c_0$, satisfying $a_0 + 2b_0 + 2c_0 \in [0, 1)$ such that

$$d(Tm, Tn) \le a_0 d(m, n) + b_0(d(m, Tm) + d(n, Tn)) + c_0(d(n, Tm) + d(m, Tn)), \quad \forall\ m, n \in M.$$

Then $T$ has a unique fixed point" [24].

The under discussed examples depict that the above results generalize BCP.

**Example 2.7.1.**

Let $M = [0, 2]$ is a nonempty set. It is a metric space with a usual metric. Consider a map $T : M \to M$ which is defined as follows:

$$T(m) = \begin{cases} 1 - 3m; & \text{if m is irrational} \in [0, 2]; \\ \dfrac{1+m}{5}; & \text{if m is rational} \in [0, 2]. \end{cases}$$

It is a Kannan mapping and its fixed point is $m_0 = \dfrac{1}{4}$.
Furthermore, this mapping $T$ is continuous at fixed point.

**Example 2.7.2.**

Let $M = [0, 2]$ be a subset of $\mathbb{R}$ accompanied with usual metric. Define $T : M \to M$ as follows:

$$T(m) = \frac{4}{5}m; \qquad \forall \quad m \in [0, 2].$$

Now take $m_1 = 0, m_2 = 2$ and we have

$$d(Tm_1, Tm_2) = \frac{8}{5}$$

and

$$d(m_1, Tm_1) + d(m_2, Tm_2) = 0 + \frac{2}{5} = \frac{2}{5}.$$

So, we have

$$d(Tm_1, Tm_2) > d(m_1, Tm_2) + d(m_2, Tm_1);\ \forall\ m_1, m_2 \in M.$$

So, it follows that $T$ map is not a Kannan contraction although it is a Banach contraction and its fixed point is $m_0 = 0$.
Furthermore, this mapping $T$ is continuous at fixed point.

**Example 2.7.3.**
Let $M = [0, 2]$ be a subset of $\mathbb{R}$ with usual metric. Define $T : M \to M$ as follows:

$$T(m) = \begin{cases} \dfrac{m}{5}; & \text{if } \ 0 \leq m < 2; \\ \dfrac{1}{4}; & \text{if } \quad m = 1. \end{cases}$$

So, we have

$$d(Tm_1, Tm_2) < d(m_1, Tm_2) + d(m_2, Tm_1);\ \forall\ m_1, m_2 \in M.$$

It can be seen that it satisfies Chatterjea contraction but not the Banach contraction.

**Example 2.7.4.**
Let $M = [-1, 2]$ is a nonempty subset of $\mathbb{R}$ along with $d$ as usual metric.
Now define $T : M \to M$ as follows:

$$T(m) = \frac{-4}{5}m;$$

for all $m \in [-1, 2]$.
Now take $m_1 = -1, m_2 = 2$ and we have

$$d(Tm_1, Tm_2) = \frac{12}{5};$$

and

$$d(m_1, Tm_2) + d(m_2, Tm_1) = \frac{3}{5} + \frac{6}{5} = \frac{9}{5}; \quad \forall\ m_1, m_2 \in M.$$

So, we have

$$d(Tm_1, Tm_2) > d(m_1, Tm_2) + d(m_2, Tm_1);\ \forall\ m_1, m_2 \in M.$$

So, it follows that $T$ map is not a Chatterjea contraction although it is a Banach contraction and its fixed point is $m_0 = 0$.
Furthermore, this mapping $T$ is continuous at fixed point.

The next presented result is given by Prudhvi [111] on $S$-metric space in 2015.

**Theorem 2.7.9.**
"Let $T$ be a self map on a complete $S$-metrics space $(M, S)$ and

$$S(Tm_1, Tm_1, Tm_2) \leq \alpha S(m_1, m_1, m_2) + \beta[S(Tm_1, Tm_1, m_1) + S(Tm_2, Tm_2, m_2)]$$

for some $\alpha, \beta \geq 0$ such that $\alpha + 2\beta < 1$ for all $m_1, m_2 \in M$. Then $T$ has a unique fixed point in $M$. Moreover, if $2\beta < 1$, then $T$ is continuous at fixed point" [111].

The following prominent result is proved by Kir *et al.* [133] on $b$-metric space in 2013.

**Theorem 2.7.10.**
"Let $(M, d)$ be a complete $b$-metric space with constant $b \geq 1$, such that $b$-metric is a continuous functional. Let $T : M \to M$ be a contraction having contraction constant $\xi \in [0, 1)$ such that $b\xi < 1$. Then $T$ has a unique fixed point" [133].

After the introduction of EbMS by Kamran *et al.* [70] in 2017, numerous extensions of fixed point results are done on such metric spaces. For the results proved by Anwar *et al.* [7, 102, 103] in 2020, it is important to include here the following result.

**Theorem 2.7.11.**
"Let $(M, d_b)$ be a complete extended $b$-metrics space such that $d_\theta$ is a continuous functional. Let $T : M \to M$ satisfy:

$$d_\theta(Tm_1, Tm_2) \leq kd_\theta(m_1, m_2)$$

for all $m_1, m_2 \in M$, where $k \in [0,1)$ be such that for each $m_0 \in M$,

$$\lim_{n,t \to 0} \theta(m_n, m_t) < \frac{1}{k},$$

here $m_n = T^n m_0, n = 0, 1, 2, ....$

Then $T$ has precisely one fixed point $\xi$.

Moreover for each $y \in M, T^n y \to \xi$" [70].

**Theorem 2.7.12.**

"Let $(M, S_b)$ be a complete $S_b$-metrics space and $T$ be a continuous self mapping on $M$ which satisfies

$$S_b(Tm_1, Tm_2, Tm_3) \leq \psi(S_b(m_1, m_2, m_3)), \quad \text{for all,} \quad m_1, m_2, m_3 \in M,$$

where $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\lim_{t \to \infty} \psi^t(n) = 0$ for each fixed $n > 0$. Then $T$ has a unique fixed point $m_0$ in $M$" [101].

The following proved theorems in uniform spaces are stated as follow:

**Theorem 2.7.13.**

"Let $(M,\mathcal{U})$ be a $S$-complete Hausdorff uniform space such that $p$ be an $E$-distance on $M$. Let $T\colon M \to M$ be an $(\alpha,\psi)$-contractive mapping and satisfying the following conditions:

1. $T$ is $\alpha$-admissible.
2. There exists $m_0 \in M$ such that $\alpha(m_0, Tm_0) \geq 1$ and $\alpha(Tm_0, m_0) \geq 1$.
3. $T$ is $p$-continuous.

Then $T$ has a fixed point $m_0 \in M$" [48].

**Theorem 2.7.14.**

"Let $(M,\mathcal{U})$ be a $S$-complete Hausdorff uniform space such that $p$ be an $E$-distance on $M$. Suppose that the pair of two $T, S\colon M \to M$ is an $(\alpha,\psi)$-contractive pair satisfying the following conditions:

1. $(T, S)$ is $\alpha$-admissible.

2. There exists $m_0 \in M$ such that $\alpha(m_0, Tm_0) \geq 1$ and $\alpha(Tm_0, m_0) \geq 1$.

3. For any sequence $\{m_t\}$ in $M$ with $m_t \to 0$ as limit $t \to \infty$ and $\alpha(m_t, m_{t+1}) \geq 1$ for each $t \in N \cup \{0\}$, then $\alpha(m_t, m) \geq 1$ for each $t \in N \cup \{0\}$.

Then $T$ and $S$ have a common fixed point" [48].

Nadler [83] extended Banach contraction principle for contraction from complete metric space $M$ into the space of all nonempty closed and bounded subsets of $M$. Suzuki [52] proved that Mizoguchi-Takahashi's fixed point theorem is indeed a real generalization of Nadler's fixed point Theorem [83].
To describe Nadler's fixed point theorem following lemmas are necessary.

**Lemma 2.7.15.**
"Let $A, B$ are sets in $CB(M)$ and for any $\epsilon > 0$ with $H(A, B) < \epsilon$, then for every $a \in A$, there exists an element $b \in B$, such that the following inequality holds: $d(a, b) < \epsilon$" [109].

**Lemma 2.7.16.**
"Let $A, B$ are sets in $CB(M)$ and for every $a \in A$, the following inequality holds: $d(a, B) \leq H(A, B)$" [109].

**Theorem 2.7.17.**
"Let $(M, d)$ be complete metric space and $T$ is a mapping from $M$ into $CB(M)$ such that, $H(Tm, Tn) \leq kd(m, n)$ for all $m, n \in M$; where $0 \leq k < 1$. Then $T$ has a fixed point" [83].

## 2.8 $F$-Mappings and $F$-Contractions

The idea of the $F$-mappings and $F$-contractions was presented by Wardowski [95] in the year 2012. He proved some extensions of Banach's result in the setting of $F$-contractions.

To reach the goals of the dissertation, the concept of $F$-contractions and fixed point results developed on such mappings are given in this section.

**Definition 2.8.1.**

"An $F$-mapping is a strictly increasing function $F\colon \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

1. For all $m_1, m_2 \in \mathbb{R}^+$, such that $m_1 < m_2$, $F(m_1) < F(m_2)$,
2. For each sequence $\{m_t\}$ of positive numbers,

$$\lim_{t\to\infty} m_t = 0 \quad \text{if and only if} \quad \lim_{t\to\infty} F(m_t) = -\infty,$$

3. There is a real number $c \in (0,1)$ such as

$$\lim_{m\to\ 0^+} m^c F(m) = 0\text{"} \quad [95].$$

**Example 2.8.1.**

The following are some examples of $F$-mapping with $c \in (0,1)$ and $m \in \mathbb{R}^+$:

1. $F(m) = -\dfrac{1}{\sqrt{m}}$ and $m > 0$.
2. $F(m) = \ln m + m$ and $m > 0$.
3. $F(m) = \ln m$ and $m > 0$.
4. $F(m) = \ln(m^2 + m)$ and $m > 0$.

The family of $F$-functions is denoted by $\mathcal{F}$.

In above examples, one can easily verify that these are examples of $F$-mapping as all the conditions of $F$-mapping for any constant $c \in (0,1)$ are hold.

**Definition 2.8.2.**

"A mapping $T : M \to M$ is said to be an $F$-contraction if there exists $\tau > 0$ such that for all $m_1, m_2 \in M$,

$$d(Tm_1, Tm_2) > 0 \Rightarrow \tau + F(d(Tm_1, Tm_2)) \leq F(d(m_1, m_2))\text{"} \quad [95]. \tag{2.1}$$

It is an important to mention here that $F$-contractions are continuous which is evident from the second property of $F$-mapping.
The famous Wardowski [95] result proved on $F$-mapping by using $F$-contraction is presented below:

**Theorem 2.8.3.**
"Let $(M, d)$ be a complete metric space and let $T : M \to M$ be an $F$-contraction. Then $T$ has a unique fixed point $m^* \in M$ and for every $m_0 \in M$ a sequence $\{T^n m_0\}_{n\in N}$ is convergent to $m^*$" [95].

Some examples of $F$-contractions are presented below:

**Example 2.8.2.**
A mapping $F\colon \mathbb{R}^+ \to \mathbb{R}$ is an $F$-mapping, such that:

1. $F(m) = -\dfrac{1}{\sqrt{m}}$ and $m > 0$ satisfies all the conditions of $F$-mapping for any constant $c \in \left(\dfrac{1}{2}, 1\right)$ and the condition (2.1) is of the form:

$$d(Tm_1, Tm_2) \leq \frac{d(m_1, m_2)}{\left(1 + \tau\sqrt{d(m_1, m_2)}\right)^2}$$

   for all $m_1, m_2 \in M,\ Tm_1 \neq Tm_2$.

2. $F(m) = \ln m + m$ and $m > 0$ satisfies all the conditions of $F$-mapping for any constant $c \in \left(\dfrac{1}{2}, 1\right)$ and the contraction condition takes the form:

$$\frac{d(Tm_1, Tm_2)}{d(m_1, m_2)} e^{d(Tm_1, Tm_2) - d(m_1, m_2)} \leq e^{-\tau}$$

   for all $m_1, m_2 \in M,\ Tm_1 \neq Tm_2$

3. $F(m) = \ln m$ and $m > 0$ satisfies all the conditions of $F$-mapping for any constant $c \in \left(\dfrac{1}{2}, 1\right)$ and the contraction condition takes the form:

$$d(Tm_1, Tm_2) \leq e^{-\tau} d(m_1, m_2)$$

   for all $m_1, m_2 \in M,\ Tm_1 \neq Tm_2$.

4. $F(m) = \ln(m^2+m)$ and $m > 0$ satisfies all the conditions of $F$-mapping for any constant $c \in \left(\frac{1}{2}, 1\right)$ and the contraction condition takes the form:

$$\frac{d(Tm_1, Tm_2)(d(Tm_1, Tm_2) + 1)}{d(m_1, m_2)(d(m_1, m_2) + 1)} \leq e^{-\tau}$$

for all $m_1, m_2 \in M,\ Tm_1 \neq Tm_2$.

The following defined mapping:

$$F(m) = -\frac{1}{\sqrt{m}} \quad \text{and} \quad m > 0$$

is an $F$-mapping and $F$-contraction that can be easily proved.

**Definition 2.8.4.**

"Consider the mappings $T\colon M \to M$ and $\alpha\colon M \times M \to [0, \infty)$. $T$ is an $\alpha$-admissible if for all $m_1, m_2 \in M$, we have $\alpha(m_1, m_2) \geq 1$
$\Rightarrow \alpha(Tm_1, Tm_2) \geq 1$" [93].

**Example 2.8.3.**

Consider $M = [0, \infty)$. Now define $T : M \to CL(M)$ as

$$Tm = \begin{cases} \left[0, \frac{m}{6}\right]; & \text{if } 0 \leq m < 6, \\ \{1\}; & \text{if } m = 6, \\ [m, m^2]; & \text{if } m > 6, \end{cases}$$

and $\alpha : M \times M \to [0, \infty)$ as

$$\alpha(m, n) = \begin{cases} 1; & \text{if } \quad m, n \in [0, 6], \\ 0; & \text{otherwise,} \end{cases}$$

clearly $T$ is an $\alpha$-admissible mapping.

Hussain *et al.* [96] presented the following result by using $F$-contraction.

**Theorem 2.8.5.**
"Let $(M, d)$ be a complete metric space. Let $T : M \to CB(M)$ satisfying the following assertions:

1. $T$ is an $\alpha_*$-admissible mapping.
2. $T$ is an $\alpha_* - \tau - F$-contraction.
3. There exists $m_0 \in M$ such that $\alpha_*(m_0, Tm_0) \geq 1$.
4. $\liminf_{s \to t^+} T(s) > 0;$ for all $t \geq 0$.
5. $T$ is a continuous mapping,

then $T$ has a fixed point in $M$" [96].

After the introduction of $F$-contraction by Wardowski [95] in 2012, different authors established many interesting results in this setting. In this perspective Secelean [9], Piri *et al.* [98], Consentino *et al.* [134], Hussain *et al.* [96] and Sgroi *et al.* [5] used $F$-contraction for different contraction conditions. Some more extensions can be seen in literature, for examples, [7, 8, 48, 96, 97, 102–104, 135].
The concept of $(\alpha, \psi)$-contraction is also utilized by Ali *et al.* [94] with the structure of uniform spaces. Stepping forward another important generalization of metric [70] appeared in literature in which author extended different fixed point results on the structure of extended $b$-metric spaces.
The following result of Ali *et al.* [75] was a source of inspiration for the current research.

**Theorem 2.8.6.**
"Let $(M, d)$ be a metric space. Let $S$ be a nonempty subset of $M$ which is complete induced with the metric $d$. Let $D : S \to CL(M)$ be a strictly $(\alpha, F)$-contractive type mapping on $S$, then $D$ has a fixed point $m \in S$ if the conditions given below are satisfied:

1. $D$ is an $\alpha$-admissible mapping.
2. There exists $m_0 \in S$ and $m_1 \in Dm_0 \cap S$ such that $\alpha(m_0, m_1) \geq 1$.
3. $D$ is a continuous mapping" [75].

# Chapter 3

# F P Theorem Via $(\alpha,F)$-Contraction NS Approach in MS

In this chapter, notion of $(\alpha,F)$-contractive nonself multivalued mappings has been introduced. The existence of the fixed point and its uniqueness on $(\alpha,F)$-contractive mapping by using the proposed notion are established. Some fixed point new results have been produced by using the notion of $(\alpha,F)$-admissible pairs. The chapter is furnished with some examples to support this new approach. Many authors produced fixed point results on nonself mappings [136–144]. The results presented in present chapter are based on the ideas given by Samet *et al.* [90], Ali *et al.* [94] and Hussain and Iqbal [100].

## 3.1 Wardowski Type $(\alpha,F)$-Contractive Approach

This section includes some fundamental definitions and some results supported by valid examples. The notion of nonself $\alpha$-admissible mapping is modified by Ali *et al.* [75] as follows:

**Definition 3.1.1.**

Consider a complete metric space $(M, d)$ for a nonempty set of $M$. Let $S$ be a nonempty subset of $M$, then a nonself mapping $D : S \to CL(M)$ is said to be an $\alpha$-admissible mapping if there exists a mapping $\alpha\colon S \times S \to [0, \infty)$ such that

$$\alpha(m_1, m_2) \geq 1 \quad \text{implies that} \quad \alpha(a, b) \geq 1$$

for each $a \in Dm_1 \cap S$ and $b \in Dm_2 \cap S$ for all $m_1, m_2 \in S$ [75].

Here an ($\alpha$,$F$)-contractive mapping is defined with a new approach of maximum distance.

**Definition 3.1.2.**

Consider a complete metric space $(M, d)$, with $M$ is a nonempty set. Also assume that $S$ be a nonempty subset of $M$. A nonself mapping $D : S \to CL(M)$ is said to be an ($\alpha$,$F$)-contractive mapping if for a function $\alpha\colon S \times S \to [0, \infty)$, an $F$-mapping ($F \in \mathcal{F}$ 2.8.1) and $\tau > 0$, the following conditions are satisfied:

1. $Dm \cap S \neq \phi$ for all $m \in S$.
2. For each $m_1, m_2 \in S$, we have

$$\tau + F\left(\alpha(m_1, m_2)H(Dm_1 \cap S, Dm_2 \cap S)\right) \leq F(M_x(m_1, m_2)), \tag{3.1}$$

where

$$\min\{\alpha(m_1, m_2)H(Dm_1 \cap S, Dm_2 \cap S)), M_x(m_1, m_2)\} > 0,$$

and

$$M_x(m_1, m_2) = \max\left\{d(m_1, m_2), \frac{d(m_1, Dm_1 \cap S) + d(m_2, Dm_2 \cap S)}{2}, \right.$$

$$\left. \frac{d(m_1, Dm_2 \cap S) + d(m_2, Dm_1 \cap S)}{2}\right\}.$$

Since $F$ is a strictly increasing function, therefore $D : S \to CL(M)$ is a strictly $(\alpha,F)$-contractive type mapping on a complete sub-space $S$ of $M$.

**Theorem 3.1.3.**

Consider a metric space $(M, d)$ and a complete nonempty subset $S$ of $M$ induced with the metric $d$. Let $D : S \to CL(M)$ be a strictly $(\alpha,F)$-contractive type map on $S$, then $D$ has a fixed point $m \in S$ if the conditions given below hold:

1. $D$ is an $\alpha$-admissible mapping.
2. there exists $m_0 \in S$ and $m_1 \in Dm_0 \cap S$ such that $\alpha(m_0, m_1) \geq 1$.
3. $D$ is a continuous mapping.

*Proof.*

By the condition 2., there is $m_0 \in S$ and $m_1 \in Dm_0 \cap S$ such that

$$\alpha(m_0, m_1) \geq 1.$$

For $m_0 = m_1$, the proof is obvious.
Now suppose that $m_0 \neq m_1$.
If $m_1 \in Dm_1 \cap S$, then $m_1$ is straight forwardly a fixed point.
Suppose that $m_1 \notin Dm_1 \cap S$.
Since $D$ is a strictly $(\alpha,F)$-contractive type mapping on $S$, the following holds.

$$\begin{aligned}
&\tau + F(\alpha(m_0, m_1)H(Dm_0 \cap S, Dm_1 \cap S)) \\
&\leq F\Big(\max\Big\{d(m_0, m_1), \frac{d(m_0, Dm_0 \cap S) + d(m_1, Dm_1 \cap S)}{2}, \\
&\qquad\qquad\qquad \frac{d(m_0, Dm_1 \cap S) + d(m_1, Dm_0 \cap S)}{2}\Big\}\Big) \\
&\leq F(\max\{d(m_0, m_1), d(m_1, Dm_1)\}) \\
&\leq F(d(m_0, Dm_0)) \\
&\leq F(d(m_0, m_1)) \ \ \forall\, m_0, m_1 \in S.
\end{aligned}$$

Therefore, we have

$$\tau + F(H(Dm_1 \cap S, Dm_0 \cap S)) \leq F(d(m_1, m_0)) \ \ \forall\, m_0, m_1 \in S.$$

This implies for $m_2 \in Dm_1 \cap S$, we have

$$\tau + F(d(m_2, m_1)) \leq F(d(m_1, m_0)) \ \ \forall\, m_2, m_1 \in S.$$

Similarly

$$\begin{aligned}
\tau + F(\alpha(m_1, m_2)H(Dm_1 \cap S, Dm_2 \cap S))& \\
&\leq F\Big(\max\Big\{d(m_1, m_2), \frac{d(m_1, Dm_1 \cap S) + d(m_2, Dm_2 \cap S)}{2}, \\
&\qquad \frac{d(m_1, Dm_2 \cap S) + d(m_2, Dm_1 \cap S)}{2}\Big\}\Big) \\
&\leq F(\max\{d(m_1, m_2), d(m_2, Dm_2)\}) \\
&\leq F(d(m_1, Dm_1)) \\
&\leq F(d(m_1, m_2)) \ \ \forall\, m_1, m_2 \in S.
\end{aligned}$$

So, we have

$$\tau + F(H(Dm_2 \cap S, Dm_1 \cap S)) \leq F(d(m_2, m_1)) \ \ \forall\, m_1, m_2 \in S.$$

This implies for $m_3 \in Dm_2 \cap S$, we have

$$\tau + F(d(m_3, m_2)) \leq F(d(m_2, m_1)) \ \ \forall\, m_3, m_2 \in S.$$

Now using the $\alpha$-admissibility, we have

$$\alpha(m_0, m_1) \geq 1$$

$$\Rightarrow \alpha(m_1, m_2) \geq 1,$$

if $m_1 \in Dm_0 \cap S$ and $m_2 \in Dm_1 \cap S$.

Continuing in this way, the following can be easily claimed for $m_{t+1} \in Dm_t \cap S$,

$$\alpha(m_t, m_{t+1}) \geq 1, \quad \forall\, m_t, m_{_{t+1}} \in S \text{ and } t \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

By running iterative,

$$\tau + F(d(m_{t+1}, m_t)) \leq F(d(m_t, m_{t-1})).$$

Inductively, we have

$$F(d(m_t, m_{t+1})) \leq F(d(m_0, m_1)) - t\tau. \tag{3.3}$$

Taking limit $t \to \infty$ on both sides

$$\lim_{t\to\infty} F(d(m_t, m_{t+1})) = -\infty. \tag{3.4}$$

From applying $F$-mapping definition, it is obtained

$$\lim_{t\to\infty} d(m_t, m_{t+1}) = 0. \tag{3.5}$$

Furthermore, denote $d(m_t, m_{t+1})$ by $d_t$. By using $F$-mapping definition, there exists $k \in (0, 1)$ such that

$$\lim_{t\to\infty} d_t^k F(d_t) = 0. \tag{3.6}$$

With the new notation, (3.3) may be expressed as

$$\begin{aligned}
F(d_t) - F(d_0) &\leq -t\tau d_t^k F(d_t) - d_t^k F(d_0) \\
&\leq d_t^k (F(d_0) - t\tau) - d_t^k F(d_0) \\
&= -t d_t^k \tau.
\end{aligned}$$

$$\Rightarrow \lim_{t\to\infty} [d_t^k F(d_t) - d_t^k F(d_0)] \leq \lim_{t\to\infty} -d_t^k t\tau.$$

$$\Rightarrow \lim_{t\to\infty} -t d_t^k \tau \geq 0.$$

$$\Rightarrow \lim_{t\to\infty} t d_t^k = 0, \ as\ \tau > 0 \text{ (using (3.5) and (3.6))}.$$

There exists $t_0 \in \mathbb{N}$ such that $td_t^k \leq 1$ for all $t \geq t_0$.

$$d_t^k \leq \frac{1}{t} \tag{3.7}$$

$$\Rightarrow \quad d_t \leq \frac{1}{t^{\frac{1}{k}}} \tag{3.8}$$

To show that $\{m_t\}$ is a Cauchy sequence, proceed as follows.

$$\begin{aligned} d(m_t, m_s) &\leq d(m_t, m_{t+1}) + d(m_{t+1}, m_{t+2}) + \ldots + d(m_{s-1}, m_s) \\ &\leq \sum_{i=t}^{\infty} d_i \\ &\leq \sum_{i=t}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned} \tag{3.9}$$

Taking limit $t \to \infty$ on both sides of (3.9),

$$\lim_{t\to\infty} d(m_t, m_s) \leq \lim_{t\to\infty} \sum_{i=t}^{\infty} \frac{1}{i^{\frac{1}{k}}} = 0.$$

So it follows that, $\{m_t\}$ is a Cauchy sequence. Therefore, it follows that $S$ is complete and there exists $m \in S$ such that

$$\lim_{t\to\infty} d(m_t, m) = 0.$$
$$\Rightarrow \lim_{t\to\infty} m_t = m.$$

Since $D$ is continuous, therefore

$$d(m, Dm) \leq \lim_{t\to\infty} H(Dm_t, Dm) = 0.$$

Thus, we have

$$d(m, Dm) = 0 \Rightarrow m \in Dm.$$

Hence $D$ has a fixed point. □

**Theorem 3.1.4.**

Consider a metric space $(M, d)$ and assume that $S$ be a complete nonempty subset of $M$ induced with metric $d$. If $D : S \to CL(M)$ be a strictly ($\alpha$,$F$)-contractive type mapping on $S$ then $D$ has a fixed point for holding the following assertions:

1. $D$ is an $\alpha$-admissible mapping.
2. there exists $m_0 \in S$ and $m_1 \in Dm_0 \cap S$ such that
$$\alpha(m_0, m_1) \geq 1.$$
3. For any sequence $\{m_t\}$ in $S$ with $m_t \to m$ and $\alpha(m_t, m_{t+1}) \geq 1\ \forall$ $t \in \mathbb{N} \cup \{0\}$, either
   (a) $\lim\limits_{t\to\infty} \alpha(m_t, m) \geq 1$ or
   (b) $\alpha(m_t, m) \geq 1$.

*Proof.*

It follows from the proof of Theorem 3.1.3 and we conclude that $\{m_t\}$ in $S$ be a Cauchy sequence such that
$$\lim_{t\to\infty} d(m_t, m) = 0,$$
and $\alpha(m_t, m_{t+1}) \geq 1$ for all $t \in \mathbb{N} \cup \{0\}$.

Assume that $d(m, Dm) \neq 0$ and using Definition 3.1.2, we obtain

$$\begin{aligned}
&\tau + F(\alpha(m_t, m)d(m_{t+1}, Dm \cap S)) \\
&\quad \leq \tau + F(\alpha(m_t, m)H(Dm_t \cap S, Dm \cap S)) \\
&\quad \leq F\left(\max\left\{d(m_t, m), \frac{d(m_t, Dm_t \cap S) + d(m, Dm \cap S)}{2},\right.\right. \\
&\qquad\qquad\qquad\qquad \left.\left.\frac{d(m_t, Dm \cap S) + d(m, Dm_t \cap S)}{2}\right\}\right).
\end{aligned}$$

Since $\alpha(m_t, m_{t+1}) \geq 1$ and $F$ is an increasing function, it is easy to observe that

$$F(d(m_{t+1}, Dm \cap S)) \leq F(d(m_0, Dm \cap S)) - t\tau.$$

Using limit $t \to \infty$, it is obtained

$$\Rightarrow \lim_{t\to\infty} F(d(m_{t+1}, Dm \cap S)) = -\infty.$$

By using the definition of $F$-mapping, we have

$$\lim_{t\to\infty} d(m_{t+1}, Dm \cap S) = 0. \tag{3.10}$$

Using assertion **3-(a)**, the following claim can be easily defended

$$\begin{aligned} d(m, Dm \cap S) &\leq \lim_{t\to\infty} \alpha(m_t, m) d(m_{t+1}, Dm \cap S) \\ &= 0. \end{aligned}$$

Furthermore, since it is obvious that

$$\begin{aligned} d(m, Dm) &\leq d(m, Dm \cap S) \\ &\leq 0. \end{aligned}$$

Therefore

$$d(m, Dm) = 0.$$

If assertion **3-(b)** is used as an argument,

$$\begin{aligned} d(m_{t+1}, Dm \cap S) &\leq \alpha(m_t, m) d(m_{t+1}, Dm \cap S) \\ &\leq \alpha(m_t, m) H(Dm_t \cap S, Dm \cap S). \end{aligned} \tag{3.11}$$

Using (3.10) one can deduce

$$d(m, Dm) \leq d(m, Dm \cap S) = 0.$$

Hence, it follows that

$$d(m, Dm) = 0.$$

That is $m \in Dm$. □

**Example 3.1.1.**

Consider $M = (-\infty, -8) \cup \left\{ \frac{1}{2^{t-1}} : t \in \mathbb{N} \right\} \cup \{0\}$, accompanied with the usual metric $d$, and $S = \left\{ \frac{1}{2^t} : t \in \mathbb{N} \right\} \cup \{0, 1\}$.

Now define $D : S \to 2^M$ on metric space as

$$Dm = \begin{cases} \left\{ \frac{1}{2^{t+1}}, 1 \right\} & \text{if} \quad m \in \left\{ \frac{1}{2^t} : t \in \mathbb{N} \right\}, \\ \{0\} & \text{if} \quad m = 0, \end{cases}$$

and $\alpha : S \times S \to [0, \infty)$ as

$$\alpha(m, n) = \begin{cases} 1 & \text{if} \quad m, n \in \left\{ \frac{1}{2^t} : t \in \mathbb{N} \right\}, \\ 0 & \text{otherwise.} \end{cases}$$

when

$$\min\{\alpha(m, n) H(Dm \cap S, Dn \cap S)), M_x(m, n)\} > 0.$$

It is clear that $Dm \cap S \neq \emptyset$ for each $m \in S$.

Let

$$F(m) = \ln m \quad \text{for all} \quad m > 0.$$

As $\alpha(m, n) \geq 1$ and $m, n \in \left\{ \frac{1}{2^t} : t \in \mathbb{N} \right\}$ so, $D$ is a multi-valued mapping. Now through the following way, $D$ can be easily seen as an ($\alpha$,$F$)-contractive and $\alpha$-admissible mapping.

Let $m = \frac{1}{2^t}$ and $n = \frac{1}{2^s}$, such that $s > t \geq 1$.

Then we have, by using the Definition 3.1.2.

$$\begin{aligned} F(\alpha(m, n) H(Dm \cap S, Dn \cap S)) - F(d(m, n)) &= \ln \left| \frac{2^{s-t} - 1}{2^{s+1}} \right| - \ln \left| \frac{2^{s-t} - 1}{2^s} \right| \\ &= \ln \frac{1}{2} \\ &< -\frac{1}{2} \ \ \forall\, m, n \in S. \end{aligned}$$

By this way, $D$ is a multi-valued ($\alpha$,$F$)-contractive mapping on $S$ with $\tau = \frac{1}{2}$. Therefore, the fixed point of $D$ is $m = 0$ as it satisfies the Theorem 3.1.3.

**Definition 3.1.5.**

Let $(M, \preceq, d)$ be an ordered metric space and $A,\ B \subseteq M$. We say that $A \prec_r B$ if for each $m \in A$ and $n \in B$, we have $m \preceq n$.

**Corollary 3.1.6.**

Consider an ordered metric space $(M, \preceq, d)$ with $(S, \preceq)$ a complete nonempty subset of $M$ induced with respect to the metric $d$. Let $D : S \to CL(M)$ be a $(\alpha,F)$-contractive mapping such that $Dm \cap S \neq \phi$ for all $m \in S$ with $m \preceq n$, then we have

$$\tau + F(\alpha(m,n)H(Dm \cap S, Dn \cap S)) \leq F(M_x(m,n)) \ \ \forall\, m,n \in S,$$

where

$$\min\{\alpha(m,n)H(Dm \cap S, Dn \cap S)), M_x(m,n)\} > 0,$$

and

$$M_x(m,n) = \max\left\{d(m,n), \frac{d(m, Dm \cap S) + d(n, Dn \cap S)}{2}, \frac{d(m, Dn \cap S) + d(n, Dm \cap S)}{2}\right\}$$

and $F$ is an increasing function. Here we also assume that the following assertions are satisfied:

1. there exists $m_0 \in S$ and $m_1 \in Dm_0 \cap S$ such that $m_0 \preceq m_1$,
2. either
   (a) $D$ is continuous, or
   (b) for any sequence $\{m_t\}$ in $S$ with $m_t \to u$ as $t \to \infty$ and $m_t \preceq m_{t+1}$ for all $t \in \mathbb{N} \cup \{0\}$, such as $t \to \infty$, $m_t \preceq u$, or
   (c) for any sequence $\{m_t\}$ in $S$ with $m_t \to u$ as $t \to \infty$ and $m_t \preceq m_{t+1}$ for all $t \in \mathbb{N} \cup \{0\}, m_t \preceq u$ for all $t \in \mathbb{N} \cup \{0\}$,

then $D$ has a fixed point $u \in S$.

*Proof.*

Define $\alpha : S \times S \to [0, \infty)$ as

$$\alpha(m,n) = \begin{cases} 1 & \text{if } m \preceq n, \\ 0 & \text{otherwise,} \end{cases}$$

We get $\alpha(m_0, m_1) = 1$, which follows from condition 1. and from the definition of $\alpha$-mapping, and from 2., we have that $m \preceq n$ implies that

$$Dm \cap S \prec_r Dn \cap S$$

and hence we get $\alpha(a, b) = 1$ implies that $\alpha(u, v) = 1$ for all $u \in Dm \cap S$ and $v \in Dn \cap S$ which follows from definitions of $\prec$ ordered metric space and $\alpha$-mapping.

Furthermore, we can easily verify that $D$ is a strictly $(\alpha,F)$-contractive type mapping on the subset $S$ of $M$. So $D$ has a fixed point as it satisfies all the conditions of previous Theorem. □

**Remark 3.1.7.**

It is worth mentioning that in Theorem 3.1.4, condition **(a)** was introduced by Samet *et al.* [90] and condition **(b)** was introduced by Ali *et al.* [75] and we have introduced these conditions for different contraction. We can verify that both conditions **(a)** and **(b)** are independent with the help of following examples.

**Example 3.1.2.**

Let $M = \left\{ \dfrac{1}{m} : m \in \mathbb{N} \right\} \cup \{0\}$. Let $m_t = \dfrac{1}{t+2}$ for all

$t \in \mathbb{N} \cup \{0\}$, then $\{m_t\}$ converges to $u^*$. Define $\alpha : M \times M \to [0, \infty)$ as

$$\alpha(m,n) = \begin{cases} \max\left\{ \dfrac{1}{m}, \dfrac{1}{n} \right\} & \text{if } m \neq 0 \ \text{ and } \ n \neq 0 \\ \dfrac{1}{m+n} & \text{if either } m = 0 \ \text{ or } \ n = 0, \\ 1 & \text{if } \ m = 0 = n. \end{cases}$$

Since

$$\begin{aligned}\alpha(m_t, m_{t+1}) &= \alpha\left(\frac{1}{t+2}, \frac{1}{t+3}\right)\\ &= t+3 > 1\end{aligned}$$

for all $t \in \mathbb{N} \cup \{0\}$,

and

$$\alpha\left(\frac{1}{t+2}, 0\right) = t+2 > 1,$$

for all $t \in \mathbb{N} \cup \{0\}$, therefore, the condition **3-(b)** of Theorem 3.1.4 is satisfied but

$$\lim_{t\to\infty} \alpha(m_t, u^*) = \lim_{t\to\infty}(t+2) = \infty,$$

which means that **3-(a)** is not satisfied.

**Example 3.1.3.**
Let $M = \left\{\frac{1}{m} : m \in \mathbb{N}\right\} \cup \{0\}$.

Let $m_t = \dfrac{1}{t+2}$ for all $t \in \mathbb{N} \cup \{0\}$, then $\{m_t\}$ converges to $u^*$.

Define $\alpha : M \times M \to [0, \infty)$ as

$$\alpha(m,n) = \begin{cases} \max\left\{\frac{1}{m}, \frac{1}{n}\right\} & \text{if } m \neq 0 \ \text{ and } \ n \neq 0 \\ \dfrac{2}{m+n+2} & \text{if either } m = 0 \ \text{ or } \ n = 0, \\ 1 & \text{if } m = 0 = n. \end{cases}$$

Since

$$\begin{aligned}\alpha(m_t, m_{t+1}) &= \alpha\left(\frac{1}{t+2}, \frac{1}{t+3}\right)\\ &= t+3 > 1\end{aligned}$$

for all $t \in \mathbb{N} \cup \{0\}$ and

$$\alpha\left(\frac{1}{t+2}, 0\right) = \frac{2t+4}{2t+5}.$$

Therefore,

$$\lim_{t\to\infty} \alpha(m_t, u^*) = \lim_{t\to\infty} \frac{2t+4}{2t+5}$$
$$= 1.$$

So, condition **3-(a)** of Theorem 3.1.4 is satisfied but in this scenario is obviously not meeting the requirement of condition **3-(b)**.

## 3.2 Applications

In this section, we are giving the solution of certain integral inclusion e. g. volterra integral inclusion via our main result. Many mathematicians worked on the solution of volterra integral inclusion via by establishing different main results and provided the application of fixed point results e. g. [5, 8, 10, 12, 69, 70, 89, 92, 97, 100, 145, 146].
For this, assume that $M = C([0,1], R)$ and $S$ be a nonempty subset of $M$ such that $S = C([0,1], R_+)$ be the space of all continuous real valued functions on $[0,1]$. $M$ is complete metric space with metric $d$ defined as

$$d(m_1, m_2) = \sup_{t\in[0,1]} |m_1(t) - m_2(t)|.$$

Now by using the Volterra-type integral inclusion as given

$$m_1(t) \in \int_0^t N(t, s, m_1(s))ds + f(t) \tag{3.12}$$

such that for all $t, s \in [0,1]$ along with the continuous functions

$$f\colon [0,1] \to \mathbb{R}_+ \quad \text{and} \quad N\colon [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}_+.$$

For each $m_1 \in C([0,1], R)$, the operator $N(t, s, m_1(s))$ is lower semi continuous. Now for the integral equation as taken above, here we define a multivalued operator $D\colon S \to CL(M)$ by as below:

$$D(m_1(t)) = \left\{ u \in C([0,1], \mathbb{R}) : u \in \int_0^t N(t,s,m_1(s))ds + f(t), t \in [0,1] \right\}.$$

Let $m_1 \in C([0,1], R)$, and denote $N_{m_1} = N(t,s,m_1(s))$ for each $t, s \in [0,1]$. Now for $N_{m_1}\colon [0,1] \times [0,1] \to P_{cv}(\mathbb{R}_+)$, by Michael's selection Theorem, there exists a continuous operator

$$n_{m_1}\colon [0,1] \times [0,1] \to \mathbb{R}_+$$

such as $n_{m_1}(t,s) \in N_{m_1}(t,s)$ for all $t, s \in [0,1]$. This shows clearly that

$$\int_0^t n_{m_1}(t,s)ds + f(t) \in D(m_1(t)).$$

Thus,it clearly follow that the operator $Dm_1$ is nonempty, and operator $Dm_1$ is closed. If multivalued operator $D$ having a fixed point, then $m_1 \in Dm_1$.

**Theorem 3.2.1.**

Let $M = C([0,1], R)$ and $S$ be a nonempty subset of $M$ such that $S = C([0,1], R_+)$ be the space of all continuous real valued functions on $[0,1]$. Let $D\colon S \to CL(M)$ be a multivalued operator defined by as below:

$$D(m_1(t)) = \left\{ u \in C([0,1], \mathbb{R}) : u \in \int_0^t N(t,s,m_1(s))ds + f(t), t \in [0,1] \right\}.$$

with the continuous functions $f\colon [0,1] \to \mathbb{R}_+$ and a multivalued function

$$N\colon [0,1] \times [0,1] \times \mathbb{R} \to P_{cv}(\mathbb{R}_+)$$

are such as for each $m_1 \in C([0,1], R)$, the operator $N(t,s,m_1(s))$ is lower semi continuous.

Suppose that the assertions given below are satisfied:

(i) there exists a continuous mapping $p\colon S \to [0,\infty)$ such as

$$(N(t,s,m_1(s)) - N(t,s,m_2(s))) \leq p(s)|m_1(s) - m_2(s))|$$

for each $t, s \in [0, 1]$ and for all $m_1, m_2 \in S$;

(ii) there exists $\tau > 0$ and $\alpha\colon S \times S \to [0, \infty)$ for each $m_1, m_2 \in S$, we have

$$\int_0^t p(s)ds \leq \frac{e^{-\tau}}{\alpha(m_1, m_2)}, \quad t \in [0, 1];$$

(iii) there exists $m_0 \in S$ and $m_1 \in Dm_0 \cap S$ with $\alpha(m_1, m_2) \geq 1$;

(iv) If $m_1 \in S$ and $m_2 \in Dm_1 \cap S$ such that $\alpha(m_1, m_2) \geq 1$, then we have $\alpha(m_2, m_3) \geq 1$ for each $m_3 \in Dm_2 \cap S$;

(v) for any sequence $m_a \in S$ such that $m_a \to u$ as $a \to \infty$ and $\alpha(m_{a+1}, m_a) \geq 1$ for each $a \in N$, it has $\alpha(m_a, u) \geq 1$ for each $a \in N$,

then Volterra-type integral inclusion having a solution.

*Proof.*

It has to show that the operator $D$ satisfy all conditions of the Theorem 3.1.3. To see the (3.1) , let $m_1, m_2 \in S$ such as $u \in Dm_1 \cap S$, then it has $n_{m_1}(t, s) \in N_{m_1}(t, s)$ for all $t, s \in [0, 1]$ such that $u(t) = \int_0^t N(t, s)ds + f(t)$, and on the other hand, from hypothesis (i), it ensures that there exists $v(t, s) \in N_b(t, s)$ such that

$$|n_{m_1}(t, s) - v(t, s)| \leq p(s)|m_1(s) - m_2(s)|;$$

$$\text{for all } t, s \in [0, 1] \, and \, m_1, m_2 \in S.$$

Consider the multivaued operator $D_1$ is given as

$$D_1(t, s) = N_{m_1}(t, s) \cap \{w \in \mathbb{R} : |n_{m_1}(t, s) - w| \leq p(s)|m_1(s) - m_2(s)|\};$$

$$\text{for all } t, s \in [0, 1] \text{ and } m_1 \in S.$$

As the given operator $D$ is lower semi continuous, so there exists a mapping

$$n_{m_2}\colon [0, 1] \times [0, 1] \to \mathbb{R}_+$$

such that $n_{m_2}(t,s) \in D_1(t,s)$ for all $t,s \in [0,1]$. Thus, we get

$$r(t) = \int_0^t n_{m_2}(t,s)ds + f(t) \in \int_0^t N(t,s)ds + f(t)$$

for all $t,s \in [0,1]$ and we have

$$\begin{aligned}
|u(t) - r(t)| &\leq \int_0^t |n_{m_1}(t,s) - n_{m_2}(t,s)|ds \\
&\leq \int_0^t p(s)|m_1(s) - m_2(s)|ds \\
&\leq \sup_{t\in[0,1]} |m_1(t) - m_2(t)| \int_0^t p(s)ds \\
&\leq d(m_1,m_2) \int_0^t p(s)ds \\
&\leq \frac{e^{-\tau}}{\alpha(m_1,m_2)} d(m_1,m_2) \text{ for all } t,s \in [0,1].
\end{aligned}$$

Consequently, it has that

$$\alpha(m_1,m_2)d(u,r) \leq e^{-\tau} d(m_1,m_2).$$

Now, if we replace the role of $m_1$ and $m_2$, we get that

$$\alpha(m_1,m_2)H(Dm_1,Dm_2) \leq e^{-\tau} d(m_1,m_2) \text{ for all } m_1,m_2 \in S,$$

whenever

$$\min\{\alpha(m_1,m_2)H(Dm_1 \cap S, Dm_2 \cap S)), M_x(m_1,m_2)\} > 0.$$

Now using the natural logarithm that belongs to $\Upsilon_s$, on the above mentioned inequality and doing the process of simplification, we have

$$\tau + \ln(\alpha(m_1,m_2)H(Dm_1,Dm_2)) \leq \ln(d(m_1,m_2)) \text{ for all } m_1,m_2 \in S.$$

So, $D$ is an $(\alpha,F)$-contractive mapping and

$$F(m) = \ln m;\ m > 0.$$

In this way, all conditions of Theorem 3.1.3 follows from hypothesis. Hence the mapping $D$ has a fixed point and integral inclusion has a solution. □

## 3.3 Conclusion

Wardowski [95] gave the idea of $F$-contraction and proved some fixed point results. These results generalizes the conventional Banach contraction principle. In [75], Ali *et al.* have used a new approach of contractive nonself multivalued mappings. Combining these approaches [75, 95] a new notion of ($\alpha$,$F$) nonself multivalued mappings has been introduced in this chapter. Using new concept, we established Theorem 3.1.3. By relaxing condition **3.** in Theorem 3.1.3 a new fixed point Theorem 3.1.4 is proved as well. These theorems together with the endorsing examples can be a good contributions towards fixed point theory.

# Chapter 4

# Fixed Point and Common Fixed Point Theorems in US (Uniform Spaces)

The main objective of this chapter is to introduce the notion of ($\alpha$,$F$)-contractive mapping and prove the results to find the existence of fixed point and its uniqueness for ($\alpha$,$F$)-contractive mapping in the structure of uniform spaces. There are certain results to prove the existence of common fixed point using the notion of $\alpha$-admissible pairs. Examples are also provided for verification of the theorems. A generalized structure of metric space, uniform space, remained an important aspect for establishing fixed point results along with $F$-contraction [5, 9, 48, 97–99, 113, 115–117, 134, 135, 147, 148]. Some preliminary and allied concepts of uniform spaces are given in subsections (2.6.7) to (2.6.17).

## 4.1 Fixed and Common Fixed Point Theorems

A new notion of ($\alpha$,$F$)-contractive mapping in uniform space is established in this section. Certain fixed point theorems are also provided in this setting with examples.

Now we continue this section for the objective to give certain new results with the following definition.

**Definition 4.1.1.**

Let $(M,\mathcal{U})$ be a uniform space and $p$ be an $E$-distance on $M$. A self mapping $T\colon M \to M$ is said to be $(\alpha,F)$-contractive mapping if there exists a functions $\alpha\colon M \times M \to [0,\infty)$, ($F \in \mathcal{F}$ 2.8.1) and constant $\tau > 0$ such that

$$\tau + F(\alpha(m_1, m_2)p(Tm_1, Tm_2)) \leq F(p(m_1, m_2)), \quad \forall\, m_1, m_2 \in M \tag{4.1}$$

whenever

$$\min\{\alpha(m_1, m_2)p(Tm_1, Tm_2), p(m_1, m_2)\} > 0.$$

**Theorem 4.1.2.**

Let $(M,\mathcal{U})$ be an $S$-complete Hausdorff uniform space and $p$ be an $E$-distance on $M$. Let $T\colon M \to M$ is $(\alpha,F)$-contractive mapping and it satisfies the following assertions:

1. the mapping $T$ is an $\alpha$-admissible;
2. there exists $m_0 \in M$ such that $\alpha(m_0, Tm_0) \geq 1$; and; $\alpha(Tm_0, m_0) \geq 1$;
3. the $T$ map is $p$-continuous,

then $T$ map has a fixed point.

*Proof.*

By condition **2.** there exists $m_0 \in M$ such that $\alpha(m_0, Tm_0) \geq 1$.

We define a sequence $\{m_t\}$ in $M$ by $m_{t+1} = Tm_t$ for all $t \in \mathbb{N} \cup \{0\}$.

If $m_{t_0} = m_{t_0+1}$ for some $t_0$, then $m = m_{t_0}$ is a fixed point of $T$.

Therefore, we consider that $m_t \neq m_{t+1}$ for all $t$.

As $T$ map is an $\alpha$-admissible, then

$$\alpha(m_0, m_1) = \alpha(m_0, Tm_0) \geq 1.$$

$$\Rightarrow \alpha(m_1, m_2) = \alpha(Tm_0, Tm_1) \geq 1.$$

Inductively, we have

$$\alpha(m_t, m_{t+1}) \geq 1, \quad \forall\, t \in \mathbb{N} \cup \{0\}. \tag{4.2}$$

Thus from (4.1), we have

$$\begin{aligned} p(m_{t+1}, m_t) &= p(Tm_t, Tm_{t-1}) \\ &\leq \tau + F(\alpha(m_t, m_{t-1})p(Tm_t, Tm_{t-1})) \\ &\leq F(p(m_t, m_{t-1})). \end{aligned}$$

We get that

$$\tau + F(\alpha(m_t, m_{t-1})p(Tm_t, Tm_{t-1})) \leq F(p(m_t, m_{t-1})).$$

Simply,

$$\tau + F(p(Tm_t, Tm_{t-1})) \leq F(p(m_t, m_{t-1})).$$

By running iteration, for all $t \in \mathbb{N}$, we have

$$\tau + F(p(Tm_{t-1}, Tm_{t-2})) \leq F(p(m_{t-1}, m_{t-2})).$$

It follows that

$$\tau + F(p(Tm_1, Tm_0)) \leq F(p(m_1, m_0)).$$

Further,

$$\tau + F(p(Tm_2, Tm_1)) \leq F(p(m_2, m_1)).$$

In the similar manner

$$\tau + F(p(Tm_3, Tm_2)) \leq F(p(m_3, m_2)).$$

Inductively, we have

$$t\tau + F(p(m_t, m_{t+1})) \leq F(p(m_0, m_1))$$

The inequality yields to the following

$$F(p(m_t, m_{t+1})) \leq F(p(m_0, m_1)) - t\tau. \tag{4.3}$$

Letting $t \to \infty$ in the above given inequality, we get

$$\lim_{t\to\infty} F(p(m_t, m_{t+1})) \leq \lim_{t\to\infty} [F(p(m_0, m_1)) - t\tau.$$

$$\Rightarrow \lim_{t\to\infty} F(p(m_t, m_{t+1})) = -\infty.$$

Thus by using the 2nd condition on the above equation of definition of $F$-mapping, we get

$$\lim_{t\to\infty} p(m_t, m_{t+1}) = 0. \tag{4.4}$$

For convenience we denote $p_t = p(m_t, m_{t+1})$ for each $t$.

Property $F_3$ implies that there is a number $k \in (0, 1)$ such that

$$\lim_{t\to\infty} p_t^k F(p_t) = 0. \tag{4.5}$$

From (4.3), we have

$$\begin{aligned}
F(p_t) - F(p_0) &\leq -t\tau. \\
p_t^k F(p_t) - p_t^k F(p_0) &\leq p_t^k (F(p_0) - t\tau) - p_t^k F(p_0) \\
&= -tp_t^k \tau \\
&\leq 0.
\end{aligned}$$

$$\begin{aligned}
\lim_{t\to\infty} [p_t^k F(p_t) - p_t^k F(p_0)] &\leq \lim_{t\to\infty} -p_t^k t\tau. \\
\lim_{t\to\infty} -tp_t^k \tau &\geq 0.
\end{aligned}$$

Letting $t \to \infty$ in the above inequality, we have

$$\lim_{t\to\infty} tp_t^k = 0, \quad \text{for} \quad \tau > 0 \quad \text{(using equations (4.4) and (4.5))}.$$

Then there exists $t_0 \in \mathbb{N}$ such that $tp_t^k \leq 1$ for all $t \geq t_0$.

$$\Rightarrow \; p_t^k \leq \frac{1}{t}$$

Thus, we have

$$\Rightarrow p_t \leq \frac{1}{t^{1/k}}.$$

As $p$ is $E$-distance, then by using triangular inequality for $s > t$, we have and for the purpose to show that $\{m_t\}$ is a $p$-Cauchy sequence.
Consider

$$\begin{aligned} p(m_t, m_s) &\leq p(m_t, m_{t+1}) + p(m_{t+1}, m_{t+2}) + \ldots + p(m_{s-1}, m_s) \\ &= \sum_{i=t}^{\infty} p_i - \sum_{i=s}^{\infty} p_i \\ &\leq \sum_{i=t}^{\infty} \frac{1}{i^{1/k}} - \sum_{i=s}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Letting $t, s \to \infty$ in the above given inequality, it follows

$$\lim_{t \to \infty} p(m_t, m_s) = 0.$$

Since $p$ is not symmetrical, by using the assumption

$$\alpha(Tm_0, m_0) \geq 1$$

and hypothesis of the theorem in the similar manner as mentioned above

$$\begin{aligned} p(m_s, m_t) &\leq p(m_s, m_{s+1}) + p(m_{s+1}, m_{s+2}) + \ldots + p(m_{t-1}, m_t) \\ &= \sum_{i=s}^{\infty} p_i - \sum_{i=t}^{\infty} p_i \\ &\leq \sum_{i=s}^{\infty} \frac{1}{i^{1/k}} - \sum_{i=t}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

We get

$$\lim_{t \to \infty} p(m_s, m_t) = 0.$$

Therefore, $\{m_t\}$ is a $p$-Cauchy sequence in $M$.

From $S$-completeness property of $M$ it follows that

$$\lim_{t\to\infty} p(m_t, m) = 0.$$

Further by hypothesis **3.**, we have

$$\lim_{t\to\infty} p(Tm_t, Tm) = 0.$$

that is,

$$\lim_{t\to\infty} p(m_{t+1}, Tm) = 0.$$

So, it has

$$\lim_{t\to\infty} p(m_t, m) = 0,$$

and

$$\lim_{t\to\infty} p(m_t, Tm) = 0.$$

Hence, by using Lemma 2.6.14-(a), we have $\quad Tm = m.$ □

In the next result, $p$-continuity of the mapping is replaced by another condition which is imposed on the space.

**Theorem 4.1.3.**

Let $(M,\mathcal{U})$ be an $S$-complete Hausdorff uniform space and $p$ be an $E$-distance on $M$. Let $T\colon M \to M$ is $(\alpha,F)$-contractive mapping and it satisfies the following assertions:

1. the $T$ map is $\alpha$-admissible;
2. there is $m_0 \in M$ such that $\alpha(m_0, Tm_0) \geq 1$
   and $\alpha(Tm_0, m_0) \geq 1$;
3. for some sequence $\{m_t\}$ in $M$ with $m_t \to m$ as $t \to \infty$ and $\alpha(m_t, m_{t+1}) \geq 1$ for all $t \in \mathbb{N} \cup \{0\}$, we have $\alpha(m_t, m) \geq 1$ for all $t \in \mathbb{N} \cup \{0\}$,

then $T$ has a fixed point.

*Proof.*

From the previous Theorem 4.1.2, and having that $\{m_t\}$ is a $p$-Cauchy in $M$ and

$$\alpha(m_t, Tm_{t+1}) \geq 1 \ \text{ for each } \ t \in \mathbb{N} \cup \{0\}.$$

Further, there exists $m \in M$ such that

$$\lim_{t\to\infty} p(m_t, m) = 0.$$

By hypothesis **3.**, we have

$$\alpha(m_t, m) \geq 1$$

for each $t \in \mathbb{N} \cup \{0\}$.

Thus by using (4.1) and triangular inequality of $p$ , we have

$$\begin{aligned} p(m_t, Tm) &\leq p(m_t, m_{t+1}) + p(m_{t+1}, Tm) \\ &\leq p(m_t, m_{t+1}) + F(\alpha(m_t, m)p(Tm_t, Tm)) \\ &< p(m_t, m_{t+1}) + F(p(m_t, m)). \end{aligned}$$

Letting $t \to \infty$ in the above inequality, we get

$$p(m_t, Tm) = 0.$$

So, it follows

$$\lim_{t\to\infty} p(m_t, m) = 0$$

and

$$\lim_{t\to\infty} p(m_t, Tm) = 0.$$

So, from Lemma 2.6.14-(a), it proves that

$$Tm = m.$$

$\square$

**Example 4.1.1.**
Consider that $M = \left\{\frac{1}{t} : t \in \mathbb{N}\right\} \cup \{0\}$ be a nonempty set and it is a metric space with $p$ as usual metric.
Define $\mathcal{U} = \{U_\epsilon | \epsilon > 0\}$. It can be seen that $(M,\mathcal{U})$ is a uniform space. Now define $T : M \to M$ as

$$Tm = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{3t+2} & \text{if } m = \dfrac{1}{t} : t > 1 \\ 1 & \text{if } m = 1, \end{cases}$$

and $\alpha : M \times M \to [0, \infty)$ as

$$\alpha(m, n) = \begin{cases} 1 & \text{if } m, n \in M - \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

It is very simple to verify that $T$ mapping is $(\alpha,F)$-contractive and consider that

$$F(m) = \ln m$$

for each $m > 0$ and $\tau = 1$.
For $m_0 = \frac{1}{2}$, we have

$$\alpha(m_0, Tm_0) = \alpha(Tm_0, m_0) = 1.$$

Further, for any sequence $\{m_t\}$ in $M$ with $m_t \to m$ implies that there exists a $t_0 \in \mathbb{N}$ such that

$$\alpha(m_{t-1}, m_t) = 1, \quad \text{for all,} \quad t \in \mathbb{N},$$

and

$$\alpha(m_t, m) = 1, \quad \text{for } t \geq t_0.$$

Now, we calculate the fixed points of the mapping $T$.

For the said purpose

$$\begin{aligned} T0 &= 0 \\ T1 &= 1 \\ Tc &= \frac{1}{3\frac{1}{c}+2} \\ Tc &= \frac{c}{3+2c} \\ \Rightarrow c &= \frac{c}{3+2c} \\ \Rightarrow 2c+3 &= 1 \\ \Rightarrow 2c &= -2 \\ \Rightarrow c &= -1 \end{aligned}$$

Hence, it can be followed from the above stated Theorem 4.1.3 that the given $T$ has these fixed points $-1, 0, 1$.

Now, we intend to find out the uniqueness of fixed point and for the said purpose consider the following assumption:

$(: J)$ For all $m, n \in Fix(T)$, we have $v \in M$ such that

$$\alpha(v, m) \geq 1 \text{ and } \alpha(v, n) \geq 1,$$

where, the set represented by $Fix(T)$ is containing all fixed points of $T$.

The guarantees of unique fixed point is given in the following theorem.

**Theorem 4.1.4.**

If we add the assumption $(: J)$ in the condition of Theorem 4.1.2 and Theorem 4.1.3, one can get the fixed point uniqueness of maps $T$.

*Proof.*

Now it is supposed that $a$ and $b$ are two distinct fixed point of $T$, contrary to our supposition. Following the condition $(J)$, there is $y \in M$ such that

$$\alpha(y, a) \geq 1 \text{ and } \alpha(y, b) \geq 1. \tag{4.6}$$

As $T$ map is an $\alpha$-admissible, thus we get

$$\alpha(T^t y, a) \geq 1 \text{ and } \alpha(T^t y, b) \geq 1, \text{ for all, } t \in \mathbb{N} \cup \{0\}. \tag{4.7}$$

We define the sequence $m_t \in M$ by

$$\begin{aligned} y_{t+1} &= Ty_t \\ &= T^t y_0 \end{aligned}$$

for all $t \in \mathbb{N} \cup \{0\}$

and $y_0 = y$.

Thus from (4.1), we get

$$\begin{aligned} \tau + F(p(y_{t+1}, a)) &\leq \tau + F(\alpha(y_t, a)p(Ty_t, Ta)) \\ &\leq F(p(y_t, a)), \text{ for all } t \in \mathbb{N} \cup \{0\}. \end{aligned}$$

Inductively, we get the following

$$p(y_t, a) \leq F(p(y_0, a)) - t\tau, \text{ for all } t \in \mathbb{N} \cup \{0\}.$$

Letting $t \to \infty$ in the above in-equality, we get

$$\lim_{t \to \infty} F(p(y_t, a)) = -\infty. \tag{4.8}$$

By using the property $F_2$ we reach

$$\lim_{t \to \infty} p(y_t, a) = 0. \tag{4.9}$$

Similarly, we have

$$\lim_{t \to \infty} p(y_t, b) = 0. \tag{4.10}$$

Thus by Lemma 2.6.14, we get $a = b$. $\square$

In the following definition we introduce the notion of ($\alpha$,$F$)-contractive pair for self mappings:

**Definition 4.1.5.**

Let $(M, v)$ be an $S$-complete Hausdorff uniform space and $p$ be an $E$-distance on $M$. A pair of mappings $T, S \colon M \to M$ is ($\alpha$,$F$)-contraction if there exist the function $\alpha \colon M \times M \to [0, \infty)$, ($F \in \mathcal{F}$ 2.8.1) and $\tau > 0$ such that

$$\tau + F(\alpha(m,n)\max\{p(Tm,Sn), p(Sm,Tn)\}) \leq F(p(m,n)) \tag{4.11}$$

for all, $m, n \in M$,

whenever

$$\max\{\alpha(m,n)\max\{p(Tm,Sn), p(Sm,Tn)\}, p(m,n)\} > 0.$$

**Theorem 4.1.6.**

Let ($M$,$\mathcal{U}$) be an $S$-complete Hausdorff uniform space and $p$ is an $E$-distance on $M$. Let a pair of self mappings $T, S \colon M \to M$ is ($\alpha$,$F$)-contractive which satisfies following assertions:

1. $(T, S)$ mapping pair is $\alpha$-admissible;
2. there is a point $m_0 \in M$ such that
$$\alpha(m_0, Tm_0) \geq 1 \text{ and } \alpha(Tm_0, m_0) \geq 1;$$
3. for some sequence $\{m_t\}$ in $M$ with $m_t \to 0$ as limit $t \to \infty$ and
$$\alpha(m_t, m_{t+1}) \geq 1 \text{ for each } t \in \mathbb{N} \cup \{0\}, \text{ then } \alpha(m_t, m) \geq 1 \text{ for each } t \in \mathbb{N} \cup \{0\};$$

then the pair $(T, S)$ has a common fixed point.

*Proof.*

From condition **2.**, we nave let $m_0 \in M$ such that

$$\alpha(m_0, Tm_0) \geq 1 \text{ and } \alpha(Tm_0, m_0) \geq 1.$$

As $(T, S)$ pair is $\alpha$-admissible, so take a sequence $m_t$ in $M$ such as

$$Tm_{2t} = m_{2t+1}$$

and

$$Sm_{2t+1} = m_{2t+2}$$

and

$$\alpha(m_t, m_{t+1}) \geq 1, \quad \text{and}, \quad \alpha(m_{t+1}, m_t) \geq 1$$

for all $t \in \mathbb{N} \cup \{0\}$.

From (4.11) and $t \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned}
\tau + p(m_{2t+1}, m_{2t+2}) &= \tau + \ p(Tm_{2t}, Sm_{2t+1}) \\
&\leq \alpha(m_{2t}, m_{2n+1}) \max\{p(Tm_{2t}, Sm_{2t+1}), p(Sm_{2t}, Tm_{2t+1})\} \\
&\leq \tau + F(\alpha(m_{2t}, m_{2t+1}) \max\{p(Tm_{2t}, Sm_{2t+1}), p(Sm_{2t}, Tm_{2t+1})\}) \\
&\leq \tau + F(p(m_{2t}, m_{2t+1})) \leq F(p(m_{2t}, m_{2t+1})).
\end{aligned}$$

This implies that

$$\tau + F(p(_{2t}, m_{2t+1})) \leq F(p(m_{2t}, m_{2t+1})) \tag{4.12}$$

Likewise, we get the following

$$
\begin{aligned}
p(m_{2t+2}, m_{2t+3}) = \; & p(Tm_{2t+1}, Sm_{2t+2}) \\
& \leq \alpha(m_{2t+1}, m_{2t+2}) \max\{p(Tm_{2t+1}, Sm_{2t+2}), \\
& p(Sm_{2t+1}, Tm_{2t+2})\} \\
& \leq \tau + F(\alpha(m_{2t+1}, m_{2t+2}) \max\{p(Tm_{2t+1}, Sm_{2t+2}), \\
& p(Sm_{2t+1}, Tm_{2t+2})\}) \\
& \leq \tau + F(p(m_{2t+1}, m_{2t+2})) \leq F(p(m_{2t+1}, m_{2t+2})).
\end{aligned}
$$

This implies that

$$\tau + F(p(m_{2t+1}, m_{2t+2})) \leq F(p(m_{2t+1}, m_{2t+2})). \tag{4.13}$$

Similarly,

$$\tau + F(p(m_{2t+2}, m_{2m+3})) \leq F(p(m_{2t+2}, m_{2t+3})). \tag{4.14}$$

Thus from (4.12), (4.13) and (4.14), and by running the iteration, we get

$$n\tau + F(p(m_t, m_{t+1})) \leq F(p(m_0, m_1)).$$

This inequality yields the following

$$F(p(m_t, m_{t+1})) \leq F(p(m_0, m_1)) - t\tau \tag{4.15}$$

for all $t \in \mathbb{N} \cup \{0\}$.

Taking limit $t \to \infty$ on both sides

$$\lim_{t\to\infty} F(p(m_t, m_{t+1})) \leq \lim_{t\to\infty}[F(p(m_0, m_1)) - t\tau]$$

$$\Rightarrow \lim_{t\to\infty} F(p(m_t, m_{t+1})) = -\infty.$$

By using the definition of $F$-mapping, we have

$$\lim_{t\to\infty} p(m_t, m_{t+1}) = 0. \tag{4.16}$$

Let choose $p_t = p(m_t, m_{t+1})$, by using $F$-property there exists $k \in (0, 1)$ such that equation (4.16) becomes

$$\lim_{t\to\infty} p_t^k F(p_t) = 0. \tag{4.17}$$

And equation (4.15) becomes

$$\begin{aligned} F(p_t) - F(p_0) &\leq -t\tau \\ p_t^k F(p_t) - p_t^k F(p_0) &\leq p_t^k (F(p_0) - t\tau) - p_t^k F(p_0) \\ &= -tp_t^k \tau \\ &\leq 0. \end{aligned}$$

$$\begin{aligned} \lim_{t\to\infty}[p_t^k F(p_n) - p_t^k F(p_0)] &\leq \lim_{t\to\infty} -tp_t^k \tau. \\ \lim_{t\to\infty} -tp_t^k \tau &\geq 0. \\ \lim_{t\to\infty} tp_t^k &= 0, \quad \text{for} \quad \tau > 0. \quad \text{(and using (4.16) and (4.17))} \end{aligned}$$

There exists $t_0 \in \mathbb{N}$ such that $tp_t^k \leq 1$ for all $t \geq t_0$.

$$\Rightarrow p_t^k \leq 1/t,$$

$$\Rightarrow p_t \leq \frac{1}{t^{1/k}}. \tag{4.18}$$

As $p$ is an $E$-distance, for $s > t$, and for the purpose to show that $\{m_t\}$ is a $p$-Cauchy.

Using the inequality, it follows

$$\begin{aligned} p(m_t, m_s) &\leq p(m_t, m_{t+1}) + p(m_{t+1}, m_{t+2}) + \ldots + p(m_{s-1}, m_s) \\ &= \sum_{i=t}^{\infty} p_i - \sum_{i=s}^{\infty} p_i \\ &\leq \sum_{i=t}^{\infty} \frac{1}{i^{1/k}} - \sum_{i=s}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Letting $t, s \to \infty$ in the above given inequality, we get

$$\lim_{t\to\infty} p(m_t, m_s) = 0.$$

Since $p$ is not symmetrical, by using the assumption

$$\alpha(Tm_0, m_0) \geq 1$$

and hypothesis of the theorem in the similar manner as mentioned above

$$\begin{aligned} p(m_s, m_t) &\leq p(m_s, m_{s+1}) + p(m_{s+1}, m_{s+2}) + \ldots + p(m_{t-1}, m_t) \\ &= \sum_{i=s}^{\infty} p_i - \sum_{i=t}^{\infty} p_i \\ &\leq \sum_{i=s}^{\infty} \frac{1}{i^{1/k}} - \sum_{i=t}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

We get

$$\lim_{t\to\infty} p(m_s, m_t) = 0.$$

We conclude that $\{m_t\}$ is a $p$-Cauchy sequence in $M$. By $S$-completeness of $M$, we have $y \in M$ such that $\lim_{t\to\infty} p(m_t, y) = 0$, which implies

$$\lim_{t\to\infty} Tm_{2t} = \lim_{t\to\infty} Sm_{2t+1} = y.$$

By hypothesis **3.**, we get

$$\alpha(m_t, y) \geq 1$$

for each $t \in \mathbb{N} \cup \{0\}$.

Thus by using (4.11) and triangular inequality of $p$, we have

$$\begin{aligned}
p(m_t, Ty) &\leq p(m_t, m_{2t+2}) + p(m_{2t+2}, Ty) \\
&= p(m_t, m_{2t+2}) + p(Sm_{2t+1}, Ty) \\
&\leq p(m_t, m_{2t+2}) + (\tau + F(\alpha(m_{2t+1}, y) \max\{p(Tm_{2t+1}, Sy), p(Sm_{2t+1}, Ty)\})) \\
&\leq p(m_t, m_{2t+2}) + (\tau + F(p(m_{2t+1}, y)) \\
&\leq p(m_t, m_{2t+2}) + F(p(m_{2t+1}, y).
\end{aligned}$$

Taking $t \to \infty$ on both side of the inequality, we obtain

$$p(m_t, Ty) = 0.$$

Further, we already have

$$\lim_{t \to \infty} p(m_t, y) = 0.$$

Thus by Lemma 2.6.14-(a), we have $Ty = y$.

Analogously, we prove that $Sy = y$.

Hence, $y$ is a common fixed point of

$$\begin{aligned}
Ty &= Sy \\
&= y.
\end{aligned}$$

□

**Remark 4.1.7.**

**Note**: If we change the assertion **2.** as stated below that Theorem 4.1.6 hold:

There exists $m_0 \in M$ such that

$$\alpha(m_0, Sm_0) \geq 1 \quad \text{and} \quad \alpha(Sm_0, m_0) \geq 1.$$

We can obtain the proof of the following theorem in the same way as we obtained the proofs of previous theorems.

**Theorem 4.1.8.**
Let $(M,\mathcal{U})$ be a $S$-complete Hausdorff uniform space and $p$ is $E$-distance on $M$. Let $T\colon M \to M$ be a self mapping for which there exist the functions $T\colon M\times M \to [0,\infty)$, $(F \in \mathcal{F}$ 2.8.1$)$ and constant $\tau > 0$ satisfying the following condition:

$$\tau + F(\alpha(m,n)\max\{p(Tm,n), p(m,Tn)\}) \leq F(p(m,n)) \tag{4.19}$$

for each $\quad m,n \in$ whenever

$$\max\{\alpha(m,n)\max\{p(Tm,n), p(m,Tn)\}, p(m,n)\} > 0.$$

Further assume that the following assertions hold:

1. $(T,S)$ pair is $\alpha$-admissible;
2. there is a point $m_0 \in M$ such as $\alpha(m_0, Tm_0) \geq 1$ and $\alpha(Tm_0, m_0) \geq 1$:
3. for some sequence $\{m_t\}$ in $M$ with $m_t \to m$ as limit $t \to \infty$ and $\alpha(m_t, m_{t+1}) \geq 1$ for all $t \in \mathbb{N}\cup\{0\}$, we have $\alpha(m_t, m) \geq 1$ for all $t \in \mathbb{N}\cup\{0\}$.

Then $T$ has a fixed point.

**Example 4.1.2.**
Let $M = \left\{\dfrac{1}{t} : t \in \mathbb{N}\right\} \cup \{0\}$ be a nonempty set with usual metric $p$.

Define $\mathcal{U} = \{U_\epsilon | \epsilon > 0\}$. It can be seen that $(M,\mathcal{U})$ is a uniform space. A mapping $T : M \to M$ is defined as

$$Tm = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{4t+1} & \text{if } m = \dfrac{1}{t} : t > 1 \\ \\ 1 & \text{if } m = 1, \end{cases}$$

and $S : M \to M$ as

$$Sm = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{3t+1} & \text{if } m = \dfrac{1}{t} : t > 1 \\ 1 & \text{if } m = 1, \end{cases}$$

and $\alpha : M \times M \to [0, \infty)$ as

$$\alpha(m,n) = \begin{cases} 1 & \text{if } m, n \in M - \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

It is very simple to verify that $(T, S)$ pair of mapping is an $(\alpha, F)$-contractive and

$$F(m) = \ln m$$

for each $m > 0$ and $\tau = 1$.

For $m_0 = \dfrac{1}{2}$, we have

$$\alpha(m_0, Tm_0) = \alpha(Tm_0, m_0) = 1.$$

Further, for any sequence $\{m_t\}$ in $M$ with $m_t \to m$ and

$$\alpha(m_t, m_{t-1}) = 1$$

for all $t \in \mathbb{N}$, we have

$$\alpha(m_t, m) = 1$$

for each $t \in \mathbb{N}$.

Now, we calculate the fixed points of the mapping $T$.

For the aforementioned purpose

$$\begin{aligned}
T0 &= 0\\
T1 &= 1\\
Tc &= \frac{1}{4\frac{1}{c}+1}\\
Tc &= \frac{c}{4+c}\\
\Rightarrow c &= \frac{c}{4+c}\\
\Rightarrow c+4 &= 1\\
\Rightarrow c &= -3
\end{aligned}$$

Now, we calculate the fixed points of the mapping $S$.

For the aforementioned purpose

$$\begin{aligned}
T0 &= 0\\
T1 &= 1\\
Tc &= \frac{1}{3\frac{1}{c}+1}\\
Tc &= \frac{c}{3+c}\\
\Rightarrow c &= \frac{c}{3+c}\\
\Rightarrow c+3 &= 1\\
\Rightarrow c &= -2
\end{aligned}$$

Thus it follows from Theorem 4.1.6, we say that the pair $(T, S)$ map has $0, 1$ common fixed points.

For the purpose to see the uniqueness of Common F P(T, S) mappings, we can use the following assertion:

**:I.** For every $m, n \in$ Common F P$(T, S)$, it has that $\alpha(m, n) \geq 1$ and the set

Common F P(T, S) represents the all common fixed points of pair mappings $(T, S)$.

**Theorem 4.1.9.**

By including assertion **:I.** in the condition of Theorem 4.1.6, one can get the uniqueness of common fixed point $(T, S)$ pair of mappings.

*Proof.*

Suppose in contrary that there are two different fixed points $m, n \in M$ of self mappings pair $(T, S)$. By condition **:I.**, it has that $\alpha(m, n) \geq 1$.

From previous Definition (4.11), we have

$$\begin{aligned}\tau + F(p(m, n)) &\leq \tau + F(\alpha(m, n) \max\{p(Tm, Sn), p(Sm, Tn)\}) \\ &\leq F(p(m, n))\end{aligned}$$

which is not possible for $p(m, n) > 0$.

Thus, we have $p(m, n) = 0$.

Further, we get $p(n, m) = 0$.

Therefore, Lemma 2.6.14-(a), implies that $u = v$. This is contrary to the supposition.

Thus, the self mapping pair $(T, S)$ has a unique common fixed point. □

The following results are immediately follow from our results by taking $\alpha(m, n) = 1$ for each $m, n \in M$.

**Corollary 4.1.10.** Let $(M, \mathcal{U})$ is a $S$-complete Hausdorff uniform space and $p$ is $E$-distance on $M$. Let $T \colon M \to M$ is a mapping for which there exist a mapping $(F \in \mathcal{F}$ 2.8.1$)$ and constant $\tau > 0$ satisfying the following condition:

$$\tau + F(p(Tm, Tn)) \leq F(p(m, n))$$

for each $m, n \in M$, whenever $p(m, n) > 0$.

Then $T$ has a fixed point.

**Corollary 4.1.11.** Let $(M,\mathcal{U})$ be a $S$-complete Hausdorff uniform space and $p$ is $E$-distance on $M$. Let $T\colon M \to M$ is a mapping for which there exist a mapping $(F \in \mathcal{F}$ 2.8.1) and constant $\tau > 0$ satisfying the following condition:

$$\tau + F(\max\{p(Tm,n), p(m,Tn)\}) \leq F(p(m,n)),$$

for each, $m, n \in M$, whenever

$$\max\{\max\{p(Tm,Sn), p(Sm,Tn)\}, p(m,n)\} > 0,$$

then the pair $(T,S)$ has a common fixed point.

# Chapter 5

# Fixed Point Theorems in Extended $b$-Metric Spaces

In this chapter, certain fixed point results are proved satisfying ($\alpha$,$F$)-contractive condition by using the new notion of extended $b$-metric spaces. Proved results are generalizations of many existing results in literature. These theorems are also supported by some examples.

## 5.1 Extended $b$-Metric Space

Recently Kamran *et al.* [70] introduced extended $b$-metric space and proved some fixed point results. Many results are produced in this structure [66–68, 149–151]. Some initial and related concepts along with examples are presented in section (2.6.1). To learn the topological properties of continuity, convergence and uniqueness of limit in extended $b$-metric space see [152]. We recall the following definitions and notions.

**Definition 5.1.1.**

Consider a mapping $\lambda : M \times M \to [1, \infty)$ for a nonempty set $M$. A function $d_\lambda : M \times M \to [0, \infty)$ is called an extended $b$-metric space if it satisfies for all $m_1, m_2, m_3 \in M$:

$d_\lambda 1$. $d_\lambda(m_1, m_2) \geq 0$,

$d_\lambda 2$. $d_\lambda(m_1, m_2) = 0 \Leftrightarrow m_1 = m_2$,

$d_\lambda 3$. $d_\lambda(m_1, m_2) = d_\lambda(m_2, m_1)$,

$d_\lambda 4$. $d_\lambda(m_1, m_3) \leq \lambda(m_1, m_3)[d_\lambda(m_1, m_2) + d_\lambda(m_2, m_3)]$,

then the pair $(M, d_\lambda)$ is called an extended $b$-metric space [103].

**Remark 5.1.2.**
If $\lambda(m_1, m_2) = m_1 + m_2 = b$ for $b \geq 1$, then the definition of $b$-metric space is obtained.

**Example 5.1.1.**
Let $M = \{1, 2, 3\}$. We define

$$\lambda : M \times M \to [1, \infty)$$

and $d_\lambda : M \times M \to [0, \infty)$ as follows:

$$\lambda(m_1, m_2) = m_1 + m_2 + 1.$$

Now define as follows

$$\begin{aligned} d_\lambda(1,1) &= d_\lambda(2,2) \\ &= d_\lambda(3,3) = 0. \end{aligned}$$

and

$$d_\lambda(1,2) = d_\lambda(2,1) = 80,$$

$$d_\lambda(1,3) = d_\lambda(3,1) = 1000,$$

$$d_\lambda(2,3) = d_\lambda(3,2) = 600.$$

Obviously the properties 1 and 2 are satisfied.

The property 3 is verified as follows: For $d_\lambda(3,1) = 80$.

$$
\begin{aligned}
d_\lambda(m_1, m_3) &\leq \lambda(m_1, m_3)[d_\lambda(m_1, m_2) + d_\lambda(m_2, m_3)].\\
d_\lambda(1,2) &\leq \lambda(1,2)[d_\lambda(1,3) + d_\lambda(3,2)].\\
80 &\leq 4(1000 + 600).\\
80 &\leq 6400.
\end{aligned}
$$

For $d_\lambda(3,2) = 600$.

$$
\begin{aligned}
d_\lambda(3,2) &\leq \lambda(3,2)[d_\lambda(3,1) + d_\lambda(1,2)].\\
600 &\leq 6(1000 + 80).\\
600 &\leq 6480.
\end{aligned}
$$

For $d_\lambda(3,1) = 1000$.

$$
\begin{aligned}
d_\lambda(3,1) &\leq \lambda(3,1)[d_\lambda(3,2) + d_\lambda(2,1)].\\
1000 &\leq 5(600 + 80).\\
1000 &\leq 3400.
\end{aligned}
$$

So, the property 3 is verified.

Then $(M, d_\lambda)$ is an extended $b$-metric space on $M$ [103].

**Example 5.1.2.**

Let $M = [0,1]$. We define

$$\lambda : M \times M \to [1, \infty)$$

and

$$d_\lambda : M \times M \to [0, \infty)$$

respectively as follows:

$$\lambda(m_1, m_2) = \frac{m_1 + m_2 + 2}{m_1 + m_2},$$

$$d_\lambda(m_1, m_2) = \begin{cases} \dfrac{1}{m_1 m_2}, & \text{if} \quad m_1, m_2 \in (0,1], \quad m_1 \neq m_2; \\ 0, & \text{if} \quad m_1, m_2 \in [0,1], \quad m_1 = m_2; \\ \dfrac{1}{m_1}, & \text{if} \quad m_2 = 0, \quad m_1 \in (0,1]. \end{cases}$$

It is easy to satisfy the first two properties which are trivially.

Now, in the following way, we consider the triangular inequality:

1. For $m_1, m_2, m_3 \in (0, 1]$, we have

$$d_\lambda(m_1, m_2) \leq \lambda(m_1, m_2)[d_\lambda(m_1, m_3) + d_\lambda(m_3, m_2)]$$

$$\Leftrightarrow$$

$$\frac{1}{m_1 m_2} \leq \frac{2 + m_1 + m_2}{m_1 + m_2} \times \frac{m_1 + m_2}{m_1 m_2 m_3}.$$

$$\Leftrightarrow$$

$$m_3 \leq 2 + m_1 + m_2.$$

2. For $m_1, m_2 \neq 0$ and $m_3 = 0$, we have

$$d_\lambda(m_1, m_2) \leq \lambda(m_1, m_2)[d_\lambda(m_1, 0) + d_\lambda(0, m_2)]$$

$$\Leftrightarrow$$

$$\frac{1}{m_1 m_2} \leq \frac{2 + m_1 + m_2}{m_1 + m_2} \times \frac{m_1 + m_2}{m_1 m_2}.$$

$$\Leftrightarrow$$

$$1 \leq 2 + m_1 + m_2.$$

3. For $m_1 \in (0,1], m_2 = 0$ and let $m_3 \in (0,1]$, we have

$$d_\lambda(m_1, 0) \leq \lambda(m_1, 0)[d_\lambda(m_1, m_3) + d_\lambda(m_3, 0)].$$

$$\Leftrightarrow$$

$$\frac{1}{m_1} \leq \frac{2+m_1}{m_1} \times \frac{1+m_1}{m_1 m_3}.$$

$$\Leftrightarrow$$

$$m_1 m_3 \leq (2+m_1)(1+m_1).$$

Therefore, it follows that for all $m_1, m_2, m_3 \in [0,1]$ we have

$$d_\lambda(m_1, m_3) \leq \lambda(m_1, m_3)[d_\lambda(m_1, m_2) + d_\lambda(m_2, m_3)].$$

Thus, the pair $(M, d_\lambda)$ is extended $b$-metric space [103].

The vital concepts of convergence, Cauchy sequence and completeness in extended $b$-metric space are defined as follows:

**Definition 5.1.3.**
Let $(M, d_\lambda)$ be an extended $b$-metric space:

1. A sequence $m_t$ in $M$ is said to be convergent to $m \in M$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\lambda(m_t, m) < \epsilon$ for all $t \geq N$. In this case, we write
$$\lim_{t \to \infty} m_t = m.$$

2. A sequence $m_t$ in $M$ is said to be Cauchy sequence if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that
$$d_\lambda(m_t, m_s) < \epsilon$$
for all $t, s \geq N$.

3. An extended $b$-metric space $(M, d_\lambda)$ is complete if every Cauchy sequence is convergent in $M$ [103].

## 5.2 Fixed Point Theorems in Extended $b$-Metric Spaces

In the current section, the main results along with related definitions are presented:

**Definition 5.2.1.**
Let $(M, d_\lambda)$ be an EbMS such that $d_\lambda$ is a continuous function. Let $T\colon M \to M$ be an $F$-contraction. We shall call that it is an $(\alpha,F)$-contractive mapping if there exists a function $\alpha\colon M \times M \to [0,\infty)$ and $\tau > 0$ such that

$$\tau + F(\alpha(m_1, m_2)d_\lambda(Tm_1, Tm_2)) \leq F(d_\lambda(m_1, m_2)) \text{ for all, } m_1, m_2 \in M. \quad (5.1)$$

with

$$\min\{\alpha(m_1, m_2)d_\lambda(Tm_1, Tm_2), d_\lambda(m_1, m_2)\} > 0.$$

**Theorem 5.2.2.**
Let $(M, d_\lambda)$ be an EbMS such that $d_\lambda$ is a continuous function. Let $T\colon M \to M$ be a self mapping and $T$ is an $(\alpha,F)$-contraction ($F \in \mathcal{F}$ 2.8.1) for $\alpha\colon M \times M \to [0,\infty)$. Then the self mapping $T$ has a fixed point if the following given conditions are observed for each $m_1, m_2 \in M$:

1. $T$ is $\alpha$-admissible self mapping.
2. There is a point $m_0 \in M$ such as $\alpha(m_0, Tm_0) \geq 1$ and $\alpha(Tm_0, m_0) \geq 1$.
3. $T$ is continuous self mapping.

*Proof.*
We take $m_0 \in M$ such that $\alpha(m_0, m_1) \geq 1$ which follows from hypothesis **2**. Let us take a sequence $\{m_t\} \in M$ defined as

$$m_{t+1} = Tm_t.$$

If we find that

$$m_{t_0} = m_{t_{0+1}}$$

for some $t_0$, then proof is obvious and $v = m_{t_0}$ is a fixed point.

Now take that $v \neq m_1$. If we have

$$m_1 = Tm_1,$$

then $m_1$ is a fixed point.

Suppose that $m_1 \neq m_{t_0}$ for all $T$.

As $T$ is $\alpha$-admissible, we follow that

$$\alpha(m_0, m_1) = \alpha(m_0, Tm_0) \geq 1.$$

$$\Rightarrow \alpha(m_1, m_2) = \alpha(Tm_0, Tm_1) \geq 1.$$

Following, we have

$$\alpha(m_t, m_{t+1}) \geq 1, \quad \forall\, t \in \mathbb{N} \cup \{0\}. \tag{5.2}$$

Therefore, from (5.1) and (5.2) it is obtained

$$\tau + F(\alpha(m_t, m_{t-1})d_\lambda(Tm_t, Tm_{t-1})) \leq F(d_\lambda(m_t, m_{t-1})) \;\; \forall\, m_0, m_1 \in M.$$

$$\Rightarrow \tau + F(d_\lambda(m_{t+1}, m_t)) \leq F(d_\lambda(m_t, m_{t-1})) \;\; \forall\, m_0, m_1 \in M.$$

It follows for all $t \in \mathbb{N}$, we have

$$\tau + F(d_\lambda(m_{t+2}, m_{t+1})) \leq F(d_\lambda(m_{t+1}, m_t)).$$

By running iteration, we have

$$\tau + F(d_\lambda(m_2, m_1)) \leq F(d_\lambda(m_1, m_0))$$
$$\tau + F(d_\lambda(m_3, m_2)) \leq F(d_\lambda(m_2, m_1)).$$

Inductively, we have

$$n\tau + F(d_\lambda(m_t, m_{t+1})) \leq F(d_\lambda(m_0, m_1))$$

$$F((d_\lambda(m_t, m_{t+1})) \leq F(d_\lambda(m_0, m_1)) - t\tau. \tag{5.3}$$

Applying limit $t \to \infty$ on both sides

$$\lim_{t\to\infty} F(d_\lambda(m_t, m_{t+1})) \leq \lim_{t\to\infty}[F(d_\lambda(m_0, m_1)) - t\tau]$$
$$\Rightarrow \lim_{t\to\infty} F(d_\lambda(m_t, m_{t+1})) = -\infty.$$

By using $F$-mapping definition, it is obtained

$$\lim_{t\to\infty} d_\lambda(m_t, m_{t+1}) = 0. \tag{5.4}$$

Let us choose

$$d_{\lambda,t} = d_\lambda(m_t, m_{t+1}).$$

By using $F$-mapping's 3rd property, there exists $k \in (0,1)$ such as (5.4) becomes

$$\lim_{t\to\infty} d^k_{\lambda,t} F(d_{\lambda,t}) = 0. \tag{5.5}$$

And (5.3) may be expressed as

$$F(d_{\lambda,t}) - F(d_{\lambda,0}) \leq -t\tau$$

$$d^k_t F(d_{\lambda,t}) - d^k_{\lambda,t} F(d_{\lambda,0}) \leq d^k_{\lambda,t}(F(d_{\lambda,0}) - t\tau) - d^k_{\lambda,t} F(d_{\lambda,0})$$

$$= -td^k_{\lambda,t}\tau$$
$$\leq 0$$

$$\lim_{t\to\infty}[d^k_{\lambda,t} F(d_{\lambda,t}) - d^k_{\lambda,t} F(d_{\lambda,0})] \leq \lim_{t\to\infty} -d^k_{\lambda,t} t\tau$$

$$\lim_{t\to\infty} -td^k_{\lambda,t}\tau \geq 0$$

$$\lim_{t\to\infty} td^k_{\lambda,t} = 0, \quad \text{for} \quad \tau > 0. \quad (\text{ and using (5.4) and (5.5)})$$

There exists $t_0 \in \mathbb{N}$ such that $td^k_{\lambda,t} \leq 1$ for all $t \geq t_0$

$$\Rightarrow d^k_{\lambda,t} \leq 1/t$$
$$\Rightarrow d_{\lambda,t} \leq \frac{1}{t^{\frac{1}{k}}}.$$

To show that $\{m_t\}$ is a Cauchy sequence for $s > t$, consider

$$\begin{aligned} d_\lambda(m_t, m_s) &\leq \lambda(m_t, m_s)d_\lambda(m_t, m_{t+1}) + \lambda(m_t, m_s)\lambda(m_{t+1}, m_s)d_\lambda(m_{t+1}, m_{t+2}) \\ &+ \ldots + \\ &\lambda(m_t, m_s)\lambda(m_{t+1}, m_s)\lambda(m_{t+2}, m_s)\ldots\lambda(m_{s-1}, m_s)d_\lambda(m_{s-1}, m_s). \end{aligned} \tag{5.6}$$

The series $\sum_{t=1}^{\infty}\left(\frac{1}{t^{1/k}}\right)\prod_{i=1}^{t}\lambda(m_i, m_s)$ converges by limit comparison test.
Let,

$$S = \sum_{t=1}^{\infty}\left(\frac{1}{t^{1/k}}\right)\prod_{i=1}^{t}\lambda(m_i, m_s),$$

$$S_t = \sum_{j=1}^{t}\left(\frac{1}{j^{1/k}}\right)\prod_{i=1}^{j}\lambda(m_i, m_s).$$

Thus for $s > t$, (5.6) implies:

$$d_\lambda(m_t, m_s) \leq S_{s-1} - S_t.$$

Using limit $t \to \infty$, it is concluded that $\{m_t\}$ is a Cauchy sequence. As $M$ is complete, thus, there exists $m \in M$ such that

$$\lim_{t\to\infty} d_\lambda(m_t, m) = 0$$

which implies

$$\lim_{t\to\infty} m_t = m.$$

Now, from the condition **3.** of the theorem, it follows $d_\lambda(Tm_t, Tm) = 0$, i. e., $d_\lambda(m_{t+1}, Tm) = 0$.

Therefore, it follows from the continuity of $d_\lambda$, $d_\lambda(Tm_t, m) = 0$ and $d_\lambda(m_t, Tm) = 0$.

Thus, it follows $Tm = m$. □

**Corollary 5.2.3.**

Let $(M, d_b)$ be a $b$-metric space such that $d_b$ is a continuous function. Let $T\colon M \to M$ be an $(\alpha,F)$-contractive mapping for $\alpha\colon M \times M \to [0, \infty)$. Then the self mapping $T$ has a fixed point if the following given conditions are satisfied for each $m_1, m_2 \in M$:

1. $T$ is $\alpha$-admissible self mapping.
2. There is a point $m_0 \in M$ such as $\alpha(m_0, Tm_0) \geq 1$ and $\alpha(Tm_0, m_0) \geq 1$.
3. $T$ is continuous self mapping.

*Proof.*

If we choose $\lambda(m_1, m_2) = b$ with $b \geq 1$ in Theorem 5.2.2 we will have the proof. □

**Definition 5.2.4.**

Let $T\colon M \to M$ be an $F$-contraction. For some $m_o \in M$,

$$\mathcal{O}_T(m_0) = \{m_0, Tm_0, T^2m_0, ...\}$$

be the orbit of $m_0$. A function $G$ from $M$ into the set of real numbers is said to be $T$-orbitally lower semi-continuous at $m \in M$ if $\{m_t\} \in \mathcal{O}_T(m_0)$ and $m_t \to m$ implies

$$G(m) \leq \liminf_{t\to\infty} G(m_t).$$

**Definition 5.2.5.**

A self mapping $T\colon M \to M$ is called an $(\alpha,F)$-contraction type mapping if there exists a function $\alpha\colon M \times M \to [0, \infty)$ and $\tau > 0$ such that

$$\tau + F(\alpha(b, Tb)d_\lambda(Tb, T^2b)) \leq F(d_\lambda(b, Tb)) \tag{5.7}$$

for all $b \in \mathcal{O}_T(m_0)$ and $m_0 \in M$, where

$$\lim_{t,s \to \infty} \lambda(m_t, m_s) = 0$$

as $T^t m_0 = m_t, t \in \mathbb{N}$,

then $T^t m_0 \to m \in M$ whenever $t \to \infty$.

Moreover, the self mapping $T$ has a fixed point $m$, if and only if

$$G(m_1) = d(m_1, Tm_1)$$

is $T$-orbitally lower semi continuous at point $m$, with

$$\min\{\alpha(b, Tb)\ d_\lambda(Tb, T^2 b)\ , d_\lambda(b, Tb)\} > 0.$$

**Theorem 5.2.6.**

Let $(M, d_\lambda)$ be an extended $b$-metric space and $d_\lambda$ is a continuous function. Let $T : M \to M$ be an $(\alpha,F)$-contractive type mapping such that;

1. $T$ maps is an $\alpha$-admissible self mapping.
2. there exists $m_0 \in M$ such that $\alpha(m_0, Tm_0) \geq 1$ and $\alpha(Tm_0, m_0) \geq 1$,

then $T$ has a fixed point if and only if

$$G(m) = d_\lambda(m, Tm)$$

is $T$-orbitally (L.S.C).

*Proof.*

We take $m_0 \in M$ and we define a sequence $\{m_t\}$ in such a way

$$m_0, Tm_0 = m_1, T^2 m_0 = Tm_1 = m_2, ..., T^t m_0 = m_t$$

and so on.

We have that $\alpha(m_0, m_1) \geq 1$. If we find that $m_0 = m_1$, then proof is obvious.

Now suppose that $m_0 \neq m_1$. If we have $m_1 = Tm_1$, then $m_1$ is a fixed point. Suppose that $m_1 \neq Tm_1$ and

$$\alpha(m_t, m_{t+1}) \geq 1, \text{ and, } \alpha(m_{t+1}, m_t) \geq 1, \text{ for all, } t \in \mathbb{N} \cup \{0\}.$$

Therefore, for $b = Tm_0$, it follows as

$$\tau + F(d_\lambda(Tm_1, Tm_0)) \leq F(d_\lambda(Tm_0, m_0)) \ \forall \, m_0, m_1 \in M. \tag{5.8}$$

As the self mapping $T$ is an $\alpha$-admissible, so it has

$$\alpha(m_0, m_1) = \alpha(m_0, Tm_0) \geq 1$$

$$\Rightarrow \alpha(m_1, m_2) = \alpha(Tm_0, Tm_1) \geq 1.$$

Inductively, it is obtained

$$\alpha(T^t m_0, T^{t+1} m_0) \geq 1, \quad \forall \, t \in \mathbb{N}. \tag{5.9}$$

Its t-th iteration (5.8), for all $t \in \mathbb{N}$, is obtained

$$\tau + F(d_\lambda(T^t m_0, T^{t+1} m_0)) \leq F(d_\lambda(T^{t-1} m_0, T^t m_0)).$$

It follows that

$$t\tau + F(d_\lambda(T^t m_0, T^{t+1} m_0)) \leq F(d_\lambda(m_0, Tm_0))$$

$$F(d_\lambda(T^t m_0, T^{t+1} m_0)) \leq F(d_\lambda(m_0, Tm_0)) - t\tau, \tag{5.10}$$

this implies

$$\lim_{t \to \infty} F(d_\lambda(T^t m_0, T^{t+1} m_0)) = -\infty. \tag{5.11}$$

Hence by $F$-mapping's definition, it is obtained

$$\lim_{t \to \infty} d_\lambda(T^t m_0, T^{t+1} m_0) = 0. \tag{5.12}$$

Let choose $d_{\lambda,t} = d_\lambda(T^t m_0, T^{t+1} m_0)$.

By applying $F$-mapping's 3rd property, there exists $k \in (0,1)$ such as (5.12) becomes

$$\lim_{t\to\infty} d^k_{\lambda,t} F(d_{\lambda,t}) = 0. \tag{5.13}$$

It is obtained from Theorem 5.2.2 it can be shown that $\{m_t\}$ is a Cauchy sequence in $M$ and $m_t = T^t m_0 \to m \in M$.

Since $G$ is orbitally lower semi continuous at $m \in M$ , so, it has

$$\begin{aligned} d_\lambda(m, Tm) &\leq \lim_{t\to\infty} \inf d_\lambda(T^t m_0, T^{t+1} m_0) \\ &= 0. \end{aligned}$$

Conversely, assume that $\{m_t\} \in \mathcal{O}(m_0)$ for $m_t \to m$ and

$$Tm = m,$$

then it is obtained that

$$\begin{aligned} G(m) = d(m, Tm) = 0 & \\ &\leq \lim_{t\to\infty} \inf G(m_t) \\ &\leq \lim_{t\to\infty} \inf d(T^t m_0, T^{t+1} m_0). \end{aligned}$$

Thus, we have $\quad Tm = m.$ □

**Corollary 5.2.7.**

Let $(M, d_b)$ be a $b$-metric space such that $d_b$ is a continuous function. Consider an $(\alpha,F)$-contractive type self mapping $T : M \to M$ which satisfy the following assertions:

1. $T$ maps is an $\alpha$-admissible;
2. there exists $m_0 \in M$ such that $\alpha(m_0, Tm_0) \geq 1 \quad$ and $\quad \alpha(Tm_0, m_0) \geq 1$,

then $T$ map has a fixed point if and only if $G(m) = d_b(m, Tm)$ is $T$-orbitally lower semi continuous.

*Proof.* If we choose $\lambda(m_1, m_2) = b$ with $b \geq 1$ in Theorem 5.2.2 we will have the proof. □

**Example 5.2.1.**

Let $M = [0, \infty)$. Define $\lambda : M \times M \to [1, \infty)$ and $d_\lambda : M \times M \to [0, \infty)$ respectively as follows:

$$\lambda(m_1, m_2) = m_1 + m_2 + 2,$$
$$d_\lambda(m_1, m_2) = (m_1 - m_2)^2.$$

Then $d_\lambda$ is an extended $b$-metric space on $M$.

Now define $T : M \to M$ as

$$Tm_1 = \frac{m_1}{2}.$$

It is obtained from contraction

$$\tau + F(d_\lambda(Tm_1, Tm_2)) \leq F(d_\lambda(m_1, m_2)),$$

follow from:

$$\tau + F(\frac{1}{4}(m_1 - m_2)^2) \leq F((m_1 - m_2)^2).$$

Define $\alpha : M \times M \to [0, \infty)$ as

$$\alpha(m_1, m_2) = \begin{cases} 1 & \text{if } m_1, m_2 \in M - \{1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$F(m) = \frac{-1}{\sqrt{m}}; \quad \text{for all } \ m > 0.$$

Now for any point $m_0 = \dfrac{1}{2}$, we have $\alpha(m_0, Tm_0) = \alpha(Tm_0, m_0) = 1$. For other steps, it is considered a given sequence $\{m_t\}$ in $M$, we have $m_t \to m$ as $t \to \infty$

and it is obtained that $\alpha(m_{t-1}, m_t) = 1;$ for all $t \in \mathbb{N}$ and also it is obtained that $\alpha(m_t, m) = 1;$ for all $t \in \mathbb{N}$.

Therefore, by the Theorem 5.2.2, it is concluded that $T$ has 0 fixed point.

## 5.3 Conclusion

Many distinct fixed point results are proved by taking the new idea of $F$-contraction, which was given by Wardowski [95]. In [48] Ali *et al.* gave an $(\alpha, F)$-contractive mapping which is genaralization of Wardowski contraction. In this chapter, the notion of $(\alpha,F)$-contraction is used for EbMS to prove some new fixed point results. The idea of EbMS was introduced by Kamran *et al.* [70] as generalization of $b$-metric spaces. So, the main results of this chapter are distinct generalizations of Wardowski [95] and many other concerned fixed point results [8, 12, 100].

# Chapter 6

# Fixed Point and Common Fixed Point Theorems in $S_b$-Metric Spaces

In this chapter, certain fixed point results in the setting of $S_b$-metric spaces are proved. To obtain this objective, the notion of $(\alpha, F)$-contractive pair in the structure of $S_b$-metric spaces is introduced and then the Wardowski type mappings in this structure is established. At the end, certain results on fixed points and common fixed points have also been presented with solved examples for the better understanding.

## 6.1 Fixed Point and Common Fixed Point Theorems in $S_b$-Metric Spaces

In [101], Souayah *et al.* produced some fixed point results on the platform of $S_b$-metric space. Motivated by the above mentioned work, some fixed point results on $(\alpha, F)$-contractive mapping on the platform of $S_b$-metric space have been reported. In this section, we have considered $(\alpha,F)$-contraction in the setting of $S_b$-metric space to produce certain fixed point results. Furthermore, an example is provided with the motive to clarify the main results. The produced results on $S$-metric space and $S_b$-metric space are given in [101, 110–112, 153–155]. Our results generalize the several existing results given by many authors in literature. The related concepts are given in sections (2.6.2) and (2.6.3). To give the main results of this section, we first present a new definition in our setting.

**Definition 6.1.1.**

Let $(M, S_b)$ be an $S_b$-metric space with $S_b$ a continuous function and $T\colon M \to M$ be a self mapping and $\alpha\colon M \times M \to [0, \infty)$, then $T$ is called $(\alpha, F, S_b)$-contractive mapping for some constant $\tau > 0$ such that it satisfies the following assertion for each $m, n \in M$:

$$\tau + F((\alpha(m,n)S_b(Tm, Tm, Tn)) \leq F(S_b(m, m, n)) \tag{6.1}$$

with

$$\min\{\alpha(m,n)S_b(Tm, Tm, Tn)), S_b(m, m, n)\} > 0.$$

**Theorem 6.1.2.**

Let $(M, S_b)$ be an $S_b$-metric space with $S_b$ a continuous function and $T\colon M \to M$ be an $(\alpha, F)$-contractive mapping for $\tau > 0$ such that it satisfies the following conditions for each $m, n \in M$:

1. The self mapping $T$ is an $\alpha$-admissible,
2. There is $m_0 \in M$ such as $\alpha(m_0, Tm_0) \geq 1$ and $\alpha(Tm_0, m_0) \geq 1$,
3. The self mapping $T$ is continuous, then $T$ has a fixed point $w$.

*Proof.*

We take $m_0 \in M$ and $m_1 \in Tm_0$ such that $\alpha(m_0, m_1) \geq 1$.

If we find that $m_0 = m_1$, then proof is obvious.

Now suppose that

$$m_0 \neq m_1.$$

If we have

$$m_1 = Tm_1,$$

then $m_1$ is a fixed point. Suppose that

$$m_1 \neq Tm_1$$

and

$$\alpha(m_t, m_{t+1}) \geq 1$$

and

$$\alpha(m_{t+1}, m_t) \geq 1, \text{ for all, } t \in \mathbb{N} \cup \{0\}.$$

From the Definition 6.1.1, for all $t \in \mathbb{N}$, we have

$$\begin{aligned}\tau + F(S_b(m_{t+1}, m_{t+1}, m_{t+2})) &= \tau + F(S_b(Tm_t, Tm_t, Tm_{t+1})) \\ &\leq \tau + F(\alpha(m_{t+1}, m_t)S_b(Tm_t, Tm_t, Tm_{t+1})) \\ &\leq F(S_b(m_t, m_t, m_{t+1})).\end{aligned}$$

We know that for 1-st iteration

$$\begin{aligned}\tau + F(S_b(m_1, m_1, m_2)) =&\tau + F(S_b(Tm_0, Tm_0, Tm_1)) \\ &\leq \tau + F(\alpha(m_1, m_0)S_b(Tm_0, Tm_0, Tm_1)) \\ &\leq F(S_b(m_0, m_0, m_1)).\end{aligned}$$

We can express it simply as:

$$\tau + F(S_b(m_1, m_1, m_2)) \leq F(S_b(m_0, m_0, m_1)) \;\; \forall\, m_0, m_1, m_2, \in M.$$

Likewise for 2-nd iteration, we have as:

$$\begin{aligned}\tau + F(S_b(m_2, m_2, m_3)) =&\tau + F(S_b(Tm_1, Tm_1, Tm_2)) \\ &\leq \tau + F(\alpha(m_2, m_1)S_b(Tm_1, Tm_1, Tm_2)) \\ &\leq F(S_b(m_1, m_1, m_2)).\end{aligned}$$

We can express it simply as:

$$\tau + F(S_b(m_2, m_2, m_3)) \leq F(S_b(m_1, m_1, m_2)) \ \ \forall\, m_1, m_2, m_3, \in M.$$

As the mapping $T$ is $\alpha$-admissible, so we have as

$$\alpha(m_0, m_1) = \alpha(m_0, Tm_0) \geq 1.$$

$$\Rightarrow \alpha(m_1, m_2) = \alpha(Tm_0, Tm_1) \geq 1.$$

Consequently, we have as under

$$\alpha(m_t, m_{t+1}) \geq 1, \quad \forall\, t \in \mathbb{N} \cup \{0\} \tag{6.2}$$

Therefore, we have inductively, for a certain iteration $t$

$$t\tau + F(S_b(m_{t+1}, m_{t+1}, m_{t+2})) \leq F(S_b(m_0, m_0, m_1))$$

and

$$F(S_b(m_{t+1}, m_{t+1}, m_{t+2})) \leq F(S_b(m_0, m_0, m_1)) - t\tau. \tag{6.3}$$

$$\Rightarrow \lim_{t\to\infty} F(S_b(m_{t+1}, m_{t+1}, m_{t+2})) \leq \lim_{t\to\infty} [F(S_b(m_0, m_0, m_1)) - t\tau]$$
$$\Rightarrow \lim_{t\to\infty} F(S_b(m_{t+1}, m_{t+1}, m_{t+2})) = -\infty.$$

Then by **(ii)** of $F$-mapping, we have

$$\lim_{t\to\infty} S_b(m_{t+1}, m_{t+1}, m_{t+2}) = 0. \tag{6.4}$$

Let's define

$$S_{b,t} = S_b(m_{t+1}, m_{t+1}, m_{t+2}).$$

By using **(iii)** of $F$-mapping there exists $k \in (0, 1)$ such as (6.4) becomes

$$\lim_{t\to\infty} S_{b,t}^k F(S_{b,t}) = 0. \tag{6.5}$$

Using the above notation, (6.3) may be expressed as:

$$F(S_{b,t}) - F(S_{b,0}) \leq -t\tau.$$

Using the (**iii**), (6.3) can be expressed as:

$$\begin{aligned}
S_{b,t}^k F(S_{b,t}) - S_{b,t}^k F(S_{b,0}) &\leq -S_{b,t}^k t\tau \\
\Rightarrow \lim_{t\to\infty}[S_t^k F(S_{b,t}) - S_{b,t}^k F(S_{b,0})] &\leq \lim_{n\to\infty} -S_{b,t}^k t\tau \\
\Rightarrow \lim_{t\to\infty} tS_{b,t}^k \tau &= 0 \\
\Rightarrow \lim_{t\to\infty} tS_{b,t}^k &= 0, \quad as \quad \tau > 0.
\end{aligned}$$

Since

$$\lim_{t\to\infty} tS_{b,t}^k = 0,$$

there exists a $t_0 \in N$ such that

$$tS_{b,t}^k \leq 1, \quad \text{for all,} \quad t \geq t_0.$$

$$\begin{aligned}
\Rightarrow S_{b,t}^k &\leq 1/t \\
\Rightarrow S_{b,t} &\leq \frac{1}{t^{1/k}}.
\end{aligned}$$

To prove that $\{m_t\}$ is a Cauchy for some $s > t$, consider

$$\begin{aligned}
S_b(m_t, m_t, m_s) &\leq b[S_b(m_t, m_t, m_s) + S_b(m_t, m_t, m_s) + S_b(m_s, m_s, m_t)] \\
&\leq b[2S_{b,t} + S_{b,s}] \\
&\leq b[2\frac{1}{t^{\frac{1}{k}}} + \frac{1}{s^{\frac{1}{k}}}].
\end{aligned} \tag{6.6}$$

Taking limit $t, s \to \infty$ on both sides and from (6.6), we get

$$\begin{aligned}
\lim_{t,s\to\infty} S_b(m_t, m_t, m_s) &\leq b[2\frac{1}{t^{1/k}} + \frac{1}{s^{1/k}}] \\
&= 0.
\end{aligned}$$

Thus, it follows that $\{m_t\}$ is an $S_b$-Cauchy sequence.

As $M$ is complete, thus, there is a point $w \in M$ such as

$$\lim_{t\to\infty} S_b(m_t, m_t, w) = 0$$

which implies

$$\lim_{t\to\infty} m_t = w.$$

Moreover, since $t$ is continuous,

$$\begin{aligned} w &= \lim_{t\to\infty} m_{t+1} \\ &= \lim_{t\to\infty} Tm_t \\ &= T(w). \end{aligned}$$

Thus, we have

$$T(w) = w.$$

□

**Example 6.1.1.**

Let

$$M = \left\{\frac{1}{t} : t \in \mathbb{N}\right\} \cup \{0\}$$

be with the usual metric $S_b$, defined as:

$$S_b(m, n, c) = |m - c| + |n - c|.$$

It can be seen easily that $(M, S_b)$ is an $S_b$ metric space.

Now consider the mapping $T : M \to M$ as

$$Tm = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{4t+2} & \text{if } m = \dfrac{1}{t} : m > 1 \\ 1 & \text{if } m = 1, \end{cases}$$

and the mapping $\alpha : M \times M \to [0, \infty)$ as under:

$$\alpha(m,n) = \begin{cases} 1 & \text{if } m, n \in M - \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

It is a very simple to find that the given mapping $T$ is an $(\alpha,F)$-contractive mapping for

$$F(m) = \ln m$$

for every $m > 0$ with some $\tau > 0$. For some $m_0 = \dfrac{1}{2}$, we have

$$\alpha(m_0, Tm_0) = \alpha(Tm_0, m_0) = 1.$$

Furthermore for any sequence $\{m_t\}$ in $M$ with $m_t \to a$ as $t \to \infty$ and

$$\alpha(m_{t-1}, m_t) = 1$$

for all $t \in \mathbb{N}$ and also we have

$$\alpha(m_t, m) = 1$$

for all $t \in \mathbb{N}$.

Now, we calculate the fixed points of the mapping $T$ As

$$\begin{aligned} T0 &= 0 \\ T1 &= 1 \\ Tm &= \frac{1}{4\frac{1}{m} + 2} \\ Tm &= \frac{m}{4 + 2m} \\ \Rightarrow m &= \frac{m}{4 + 2m} \\ \Rightarrow 2m + 4 &= 1 \\ \Rightarrow m &= \frac{-3}{2} \end{aligned}$$

So, it follows from Theorem 6.1.2 that $T$ has $0, 1, \frac{-3}{2}$ are fixed points.

In the search of uniqueness fixed point, we assume the following condition:

$(\Xi)$ : For all $m, n \in Fix(T)$, there is a point $v \in M$ such as

$$\alpha(v, m) \geq 1 \text{ and } \alpha(v, n) \geq 1.$$

Further, $Fix(T)$ represents the set of all fixed points of $T$ map.

The forthcoming result provides a guarantee of the uniqueness of the fixed point.

**Theorem 6.1.3.**

If we add the new condition $(\Xi)$ in the hypothesis of above Theorem 6.1.2, we get the unique fixed point of the self mapping $T$.

*Proof.*

Contrarily, assume that there exists two different fixed points $m$ and $n$ of $T$. Condition $(\Xi)$ implies that there exists $v \in M$ such that

$$\alpha(v, m) \geq 1 \text{ and } \alpha(v, n) \geq 1. \tag{6.7}$$

As $T$ is $\alpha$-admissible, therefore from in-equations (6.7), we get

$$\alpha(T^t v, m) \geq 1 \text{ and } \alpha(T^t v, n) \geq 1, \quad \text{for all } \ t \in \mathbb{N} \cup \{0\}. \tag{6.8}$$

Let us define a sequence $m_t \in M$ by

$$m_{t+1} = Tm_t = T^t m_0;$$

for all $\quad t \in \mathbb{N} \cup \{0\}$.

From in-equations (6.8) and (6.1), we get

$$\tau + F(S_b(m_{t+1}, m_{t+1}, m)) \leq \tau + F(\alpha(m_t, m)S_b(Tm_t, Tm_t, Tm))$$

$$\leq F(S_b(m_t, m_t, m)); \quad \text{for all } \ t \in \mathbb{N} \cup \{0\}.$$

Inductively, we have

$$F(S_b(m_t, m_t, m)) \le F(S_b(m_0, m_0, m)) - t\tau, \quad \text{for all} \quad t \in \mathbb{N} \cup \{0\}.$$
$$\Rightarrow \lim_{t\to\infty} F(S_b(m_t, m_t, m)) = -\infty. \tag{6.9}$$

It follows from the (**ii**) of $F$-mapping,

$$\lim_{t\to\infty} S_b(m_t, m_t, m) = 0. \tag{6.10}$$

Similarly

$$\lim_{t\to\infty} S_b(m_t, m_t, n) = 0, \tag{6.11}$$

and the limit of a sequence is always unique, hence $m = n$.

Hence, it has obtained that $u$ is the required unique fixed point. □

**Definition 6.1.4.** [85]

Let $(M, S_b)$ be an $S_b$-metric space with $S_b$ a continuous function and a pair of mappings $T, S\colon M \to M$ is an $\alpha$-admissible pair if for any $m, n \in M$ with $\alpha(m, n) \ge 1$, we get

$$\alpha(Tm, Sn) \ge 1 \text{ and } \alpha(Sm, Tn) \ge 1.$$

**Definition 6.1.5.**

Let $(M, S_b)$ be an $S_b$-metric space with $S_b$ a continuous function and a pair of mappings $T, S\colon M \to M$ is $(\alpha, F, S_b)$-contractive pair if there exists a function $\alpha\colon M \times M \to [0, \infty)$ and $\tau > 0$ such that

$$\tau + F(\alpha(m, n) \max\{S_b(Tm, Tm, Sn), S_b(Sm, Sm, Tn)\}) \le F(S_b(m, m, n)), \tag{6.12}$$

for each $m, n \in M$ with

$$\max\{\alpha(m, n) \max\{S_b(Tm, Tm, Sn), S_b(Sm, Sm, Tn)\}, S_b(m, m, n)\} > 0.$$

**Theorem 6.1.6.**

Let $(M, S_b)$ be an $S_b$-metric space with $S_b$ a continuous function. Consider $(\alpha, F, S_b)$-contractive pair $(T, S)$ satisfies as the following:

1. The pair $(T, S)$ is $\alpha$-admissible,
2. There is a point $m_0 \in M$ such as

$$\alpha(m_0, Tm_0) \geq 1 \text{ and } \alpha(Tm_0, m_0) \geq 1,$$

3. For any sequence $\{m_t\}$ in $M$ with $m_t \to 0$ as $t \to \infty$ and $\alpha(m_t, m_{t+1}) \geq 1$ for all $t \in \mathbb{N} \cup \{0\}$, then $\alpha(m_t, m) \geq 1$ for all $t \in \mathbb{N} \cup \{0\}$.

Then the self mappings pair $(T, S)$ has a common fixed point.

*Proof.*

For any $m_0 \in M$ and by hypothesis **2.**, we have

$$\alpha(m_0, Tm_0) \geq 1 \quad \text{and} \quad \alpha(Tm_0, m_0) \geq 1.$$

Since the pair $(T, S)$ is an $\alpha$-admissible, therefore a sequence $\{m_t\}$ in $M$ such as

$$Tm_{2t} = m_{2t+1} \quad \text{and} \quad Sm_{t+1} = m_{2t+2}$$

and we have

$$\alpha(m_t, m_{t+1}) \geq 1 \quad \text{and} \quad \alpha(m_{t+1}, m_t) \geq 1$$

for every $t \in \mathbb{N} \cup \{0\}$.

As $(T, S)$ is an $(\alpha, F, S_b)$-contractive pair, so we have

$$\begin{aligned}
\tau + F(S_b(m_{2t+1}, m_{2t+1}, m_{2t+2})) =& \tau + F(S_b(Tm_{2t}, Tm_{2t}, Sm_{2t+1})) \\
& \leq \tau + F(\alpha(m_{2t}, m_{2t+1}) \times \\
& \max\{S_b(Tm_{2t}, Tm_{2t}, Sm_{2t+1}), \\
& S_b(Sm_{2t}, Sm_{2t}, Tm_{2t+1})\} \\
& \leq F(S_b(m_{2t}, m_{2t}, m_{2t+1})).
\end{aligned}$$

This implies that

$$\tau + F(S_b(m_{2t+1}, m_{2t+1}, m_{2t+2})) \leq F(S_b(m_{2t}, m_{2t}, m_{2t+1})). \tag{6.13}$$

Similarly, we can get that

$$\begin{aligned}\tau + F(S_b(m_{2t+2}, m_{2t+2}, m_{2t+3})) =& \tau + F(S_b(Sm_{2t+1}, Sm_{2t+1}, Tm_{2t+2}))\\ &\leq \tau + F(\alpha(m_{2t+1}, m_{2t+2})\times\\ &\max\{S_b(Tm_{2t+1}, Tm_{2t+1}, Sm_{2t+2}),\\ &S_b(SM_{2t+1}, Sm_{2t+1}, Tm_{2t+2})\}\\ &\leq F(S_b(m_{2t+1}, m_{2t+1}, m_{2t+2})).\end{aligned}$$

Similarly, it implies that

$$\tau + F(S_b(m_{t+1}, m_{t+1}, m_{t+2})) \leq F(S_b(m_t, m_t, m_{t+1})). \tag{6.14}$$

Then following from (6.13) and (6.14), we have

$$F(S_b(m_t, m_t, m_{t+1})) \leq F(S_b(m_0, m_0, m_1)) - t\tau \tag{6.15}$$

$$\text{for all } \ t \in \mathbb{N} \cup \{0\}.$$

$$\Rightarrow \lim_{t\to\infty} F(S_b(m_t, m_t, m_{t+1})) \leq \lim_{t\to\infty} [F(S_b(m_0, m_0, m_1)) - t\tau].$$

$$\Rightarrow \lim_{t\to\infty} F(S_b(m_t, m_t, m_{t+1})) = -\infty.$$

$$\Rightarrow \lim_{t\to\infty} S_b(m_t, m_t, m_{t+1}) = 0. \tag{6.16}$$

Let's define $S_{b,t} = S_b(m_t, m_t, m_{t+1})$ and by using $c.$ of $F$-mapping definition there exists $k \in (0, 1)$ such that equation (6.16) becomes

$$\lim_{t\to\infty} S_{b,t}^k F(S_{b,t}) = 0. \tag{6.17}$$

The equation (6.16) becomes as follows and using (**iii**) of $F$-mapping definition and limit, it can be concluded as follows:

$$F(S_{b,t}) - F(S_{b,0}) \leq -t\tau$$
$$\Rightarrow \; S^k_{b,t}F(S_{b,t}) - S^k_{b,t}F(S_{b,0}) \leq -tS^k_{b,t}\tau$$

$$\Rightarrow \; \lim_{t\to\infty}[S^k_{b,t}F(S_{b,t}) - S^k_{b,t}F(S_{b,0})] \leq \lim_{t\to\infty} -tS^k_{b,t}\tau$$
$$\Rightarrow \; \lim_{t\to\infty} tS^k_{b,t} = 0, \quad as \quad \tau > 0.$$

So, there exists $t_0 \in \mathbb{N}$ such that

$$tS^k_{b,t} \leq 1; \;\; \text{for all}; \;\; t \geq t_0$$
$$\Rightarrow S_{b,t} \leq \frac{1}{t^{1/k}}. \tag{6.18}$$

For $s > t$, we show that $\{m_t\}$ is an $S_b$-Cauchy sequence, consider

$$\begin{aligned} S_b(m_t, m_t, m_s) &\leq b[S_b(m_t, m_t, m_l) + S_b(m_t, m_t, m_l) + S_b(m_s, m_s, m_l)] \\ &\leq b[2S_{b,t} + S_{b,s}] \\ &\leq b[2\frac{1}{t^{1/k}} + \frac{1}{s^{1/k}}]. \end{aligned} \tag{6.19}$$

Taking limit $t, s \to \infty$ on both sides and from (6.18), we get

$$\lim_{n,m\to\infty} S_b(m_t, m_t, m_s) \leq b[2\frac{1}{t^{1/k}} + \frac{1}{s^{1/k}}]$$

$$= 0.$$

Therefore,$\{m_t\}$ is a Cauchy sequence.

So, there exists an element $w \in M$ such that

$$\lim_{t\to\infty} S_b(m_t, m_t, w) = 0.$$

So, this implies

$$\lim_{t\to\infty} Tm_{2t} = \lim_{t\to\infty} Sm_{2t+1} = w.$$

From equation (6.14) and from assumption **3.** of the theorem, we get

$$\begin{aligned}
S_b(m_t, m_t, Tw) &\leq b[S_b(m_t, m_t, m_{2t+2}) + S_b(m_t, m_t, m_{2t+2}) + S_b(m_{2t+2}, m_{2t+2}, Tw)] \\
&= b[S_b(m_t, m_t, m_{2t+2}) + S_b(m_t, m_t, m_{2t+2}) + \\
&S_b(Sm_{2t+1}, Sm_{2t+1}, Tw)] \\
&\leq b[S_b(m_t, m_t, m_{2t+2}) + S_b(m_t, m_t, m_{2t+2}) + \alpha(m_{2t+1}, w)\times \\
&\max\{S_b(Tm_{2n+1}, Tm_{2n+1}, sw), S_b(Sm_{2n+1}, Sm_{2t+1}, Tw)\}))] \\
&< b[S_b(m_t, m_t, m_{2t+2}) + S_b(m_t, m_t, m_{2t+2}) + S_b(m_{2t+1}, m_{2t+1}, w)] \\
&< b[2S_{b,t} + S_{b,2t+1}] \\
&< b[2\frac{1}{t^{1/k}} + \frac{1}{(2t+1)^{1/k}}].
\end{aligned}$$

Taking limit on the both sides of above as follows

$$\lim_{t\to\infty} S_b(m_t, m_t, Tw) = 0$$

and

$$\lim_{t\to\infty} S_b(m_t, m_t, w) = 0.$$

Thus, we have $Tw = w$.

Analogously, we can find that $Sw = w$.

Hence, $Tw = Sw = w$. □

**Remark 6.1.7.**

Note that the Theorem 6.1.6 also holds if we replace condition **2.** by as given below:

There exists $m_0 \in M$ such that

$$\alpha(m_0, Sm_0) \geq 1 \text{ and } \alpha(Sm_0, m_0) \geq 1.$$

**Example 6.1.2.**

Let $(M, S_b)$ is an $S_b$-metric space where $M = \left\{\frac{1}{t} : t \in \mathbb{N}\right\} \cup \{0\}$ with metric defined as:

$$S_b(m, n, c) = |m - c| + |n - c|.$$

We can easily find that $(M, S_b)$ is an $S_b$-metric space. Assume $T : M \to M$ as defined below:

$$Tm = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{3t+1} & \text{if } m = \dfrac{1}{t} : t > 1 \\ 1 & \text{if } m = 1, \end{cases}$$

and $S : M \to M$ is defined by:

$$Sm = \begin{cases} 0 & \text{if } m = 0 \\ \dfrac{1}{4t+1} & \text{if } \ m = \dfrac{1}{t} : t > 1 \\ 1 & \text{if } m = 1, \end{cases}$$

and the mapping $\alpha : M \times M \to [0, \infty)$ is as defined below:

$$\alpha(m, n) = \begin{cases} 1 & \text{if } m, n \in M - \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

Here, we have that $(T, S)$ map is an $(\alpha$,$F)$-contractive for

$$F(m) = \ln m$$

for every $m > 0$ with some $\tau > 0$.

Here for some $m_0 = \dfrac{1}{2}$, we get that

$$\alpha(m_0, Tm_0) = \alpha(Tm_0, m_0)$$

$$= 1.$$

Furthermore for any sequence $\{m_t\}$ in $M$, we obtain

$$m_t \to m \text{ for } t \to \infty$$

and also we get

$$\alpha(m_t, m_{t+1}) \geq 1; \text{ for all; } t \in \mathbb{N} \cup \{0\}.$$

We have also

$$\alpha(m_t, m) \geq 1; \text{ for all; } t \in \mathbb{N}.$$

Now, we calculate the fixed points of the mapping $T$. So, for the said purpose

$$\begin{aligned}
T0 &= 0 \\
T1 &= 1 \\
Tc &= \frac{1}{3\frac{1}{c}+1} \\
Tc &= \frac{c}{3+c} \\
\Rightarrow c &= \frac{c}{3+c} \\
\Rightarrow c+3 &= 1 \\
\Rightarrow c &= -2
\end{aligned}$$

Now, we calculate the fixed points of the mapping $S$. So, for the said purpose

$$\begin{aligned}
T0 &= 0 \\
T1 &= 1 \\
Tc &= \frac{1}{4\frac{1}{c}+1} \\
Tc &= \frac{c}{4+c} \\
\Rightarrow c &= \frac{c}{4+c} \\
\Rightarrow c+4 &= 1 \\
\Rightarrow c &= -3
\end{aligned}$$

Thus, by Theorem 6.1.6, the pair of self-mappings ($T$,$S$) has $0, 1$ common fixed points.

To find out the unique common fixed point for the pair of self-mappings, we apply the condition given below:

($\Omega$) : For all $m, n \in$ Common F P$(T, S)$, it follows that $\alpha(m, n) \geq 1$, where as the notation Common F P of $(T, S)$ is used to represent the required set of all common fixed points for the pair of self mappings ($T$,$S$).

**Theorem 6.1.8.**

By including condition ($\Omega$) in the statement of the Theorem 6.1.6, we conceive the uniqueness of the common fixed point for the pair mappings ($T$,$S$).

*Proof.*

We consider on contrary that $m, n \in M$ are two different common fixed points of the pair mappings ($T$,$S$). Following from the given condition **1.** of Theorem 6.1.6, we have

$$\begin{aligned}\tau + F(S_b(m, m, n)) &\leq F(\alpha(m, n) \max\{S_b(Tm, Tm, Sn), S_b(Sn, Sn, Tm)\}) \\ &\leq F(S_b(m, m, n)),\end{aligned}$$

which is not possible for $S_b(m, m, n) > 0$ and as a result, we get $S_b(m, m, n) = 0$. Likewise, one can obtain that $S_b(n, n, m) = 0$. Therefore, we get $m = n$ which is contrary to our supposition. Hence $T$ and $S$ have a unique common fixed point. □

The next results obviously follow by assuming $\alpha(m, n) = 1$ for all $m, n \in M$.

**Corollary 6.1.9.**

Let $(M, S)$ be an $S$-metric space with $S$ a continuous function. Also consider $T: M \to M$ is a mapping and $F$ be a mapping ($F \in \mathcal{F}$ 2.8.1) such that

$$\tau + F(S(Tm, Tm, Tn)) \leq F(S(m, m, n)) \; \forall \; m, n \in M; \text{ where } S(m, m, n) > 0.$$

Then $T$ has a unique fixed point.

**Corollary 6.1.10.**

Let $(M, S)$ be an $S$-metric space with $S$ a continuous function. Also consider $T, S\colon M \to M$ are mappings and $F$ be a mapping ($F \in \mathcal{F}$ 2.8.1) such that

$$\tau + F(\max\{S(Tm, Tm, Sn), S(Sm, Sm, Tm)\}) \leq F(S(m, m, n)) \,\forall\, m, n \in M;$$

$$\text{where we have } \max\{\max\{S(Tm, Tm, Sn), S(Sm, Sm, Tn)\}, S(m, m, n)\} > 0.$$

Then the pair ($T$,$S$) have a unique fixed point.

**Corollary 6.1.11.**

Let $(M, S_b)$ be an $S_b$-metric space with $S_b$ a continuous function. Also consider $T\colon M \to M$ is a mapping and $F$ be a mapping ($F \in \mathcal{F}$ 2.8.1) such that

$$\tau + F(S_b(Tm, Tm, Tn)) \leq F(S_b(m, m, n)) \,\forall\, m, n \in M \text{ ; where } S_b(m, m, n) > 0.$$

Then $T$ has a unique fixed point.

**Corollary 6.1.12.**

Let $(M, S_b)$ be an $S_b$-metric space with $S_b$ a continuous function. Also consider $T, S\colon M \to M$ are mappings and $F$ be a mapping ($F \in \mathcal{F}$ 2.8.1) such that

$$\tau + F(\max\{S_b(Tm, Tm, Sn), S_b(Sn, Sn, Tm)\}) \leq F(S_b(m, m, n)) \,\forall\, m, n \in M;$$

$$\text{where we have } \max\{\max\{S_b(Tm, Tm, Sn), S_b(Sm, Sm, Tn)\}, S_b(m, m, n)\} > 0.$$

Then $T$ and $S$ have a unique fixed point.

# Chapter 7

# Conclusion and Future Work

## 7.1 Conclusion

This is a very important chapter for the conclusion of the targets that acheived in the whole dissertation. In this dissertation, the main source of inspiration is the work of Wardowski, who gave the idea of $F$-contraction [95]. This idea is generalized by introducing $(\alpha, F)$-contractive mapping. This new contractive mapping is used on different abstract spaces to produce the fixed point results.

1. In this thesis, we have generalized the idea of Wardowski types contractions by following the results proved by Ali *et al.* [75] in metric spaces for nonself multivalued contractive mappings. In [12] Kamran *et al.* proved results in $b$-metric spaces by applying $F$-contraction. By analyzing these approaches and consolidating these results, a new notion of $(\alpha,F)$-contractive for nonself multivalued mappings is introduced in the setting of metric spaces. After completing the preliminaries for such mappings, some fixed point theorems with new approach in metric spaces on nonself multivalued mappings are established. We have proved fixed point Theorem 3.1.3. By relaxing condition **3.** in Theorem 3.1.3 a new fixed point Theorem 3.1.4 is proved as well. These theorems are supported by examples. A novel application is also provided for the elaboration of applicability of these proved results.

2. A new notion of ($\alpha$,$F$)-contractive mapping is introduced in the preview of uniform spaces. For the said purpose the fixed point results that are proved by Kamran *et al.* [12] in $b$-metric spaces by using $F$-contractive mappings and results proved by Ali *et al.* [94] in uniform spaces are used as base. Combining these contributions, certain novel fixed point and common fixed point results, in this setting, are established. The examples provided, endorsed the authentication of these theorems. These results is an ambient contribution in fixed point theory.

3. $(\alpha, F)$-contraction notion is introduced on the structure of EbMS. Some different fixed point results are produced in this context which are the generalizations of many already existing results in the literature [8, 12, 100] and Wardowski [95]. These results are supported by solved examples.

4. The concept of $(\alpha, F)$-contractive mappings is also used on the platform of $S_b$-metric spaces. $S_b$-metric space was introduced by Souayah *et al.* [101]. Some fixed point results in this setting are proved. These ideas are extended to establish common fixed point and fixed point theorems by combining the ideas of $b$-metric spaces and $S$-metric spaces. Examples are given in the support of these results. Theorems proved on $S_b$-metric spaces, using the notion of $(\alpha, F)$-contractive mappings, are the generalizations of many existing results in literature.

## 7.2 Future Work

The future research work contains the generalizations that will be based on the following ideas:

1. Best proximity theorems with proximally complete metric for single valued and set-valued mappings in different spaces.

2. Using Fuzzy mappings in single and multivalued in different spaces for $(\alpha, F)$-contractive mappings.

3. Already existing results on Fuzzy mappings are planed to extend on extended $b$-metric space for $(\alpha, F)$-contractive mappings.

4. Best proximity points results can be proved on extended $b$-metric space for $(\alpha, F)$-contractive single and multivalued mappings.

Some results are submitted for possible publication.

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